Calculus MA1001-A Midterm 1

National Central University, Nov. 1, 2018

Problem 1. Complete the following.

1. (5%) Let f be defined on an open interval I containing c. State the definition of the differentiability of f at c.

2. (5%) State Rolle's Theorem.

Problem 2. (10%) Compute the following limits (without using L'Hôpital's rule in Problem 3).

(1) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{|x|}$ (2) $\displaystyle\lim_{x\to 0^+} \frac{\sqrt[3]{1+x}-1}{1-\cos\sqrt{x}}$.

Problem 3. (10%) Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} (1-\cos x)\sin(\cot x) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$
Find the derivatives of f.

Problem 4. Complete the following.

(1) (5%) Show the Cauchy mean value theorem: Let $f, g : [a, b] \to \mathbb{R}$ be continuous and f, g are differentiable on (a, b). Show that if $g(a) \neq g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f'(c)}{g'(c)}\,.$$

(2) (10%) Suppose that f is twice continuously differentiable (i.e., f'' is continuous). Prove that for $a < x < b$, there exists $c \in (a, b)$ such that

$$\frac{\dfrac{f(x)-f(a)}{x-a} - \dfrac{f(b)-f(a)}{b-a}}{x-b} = \frac{1}{2} f''(c)\,.$$

(3) (5%) Show that L'Hôpital's rule: Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, $c \in (a, b)$, and $f(c) = g(c) = 0$. Suppose that $\dfrac{f(x)}{g(x)}$ and $\dfrac{f'(x)}{g'(x)}$ are both defined on (a, b), except possibly at c. Show that if the limit $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}\,.$$

(4) (10%) Use L'Hôpital's rule to compute the the limit in Problem 1.

Problem 5. (10%) Suppose that $f : \mathbb{R} \to (0, \infty)$ and $g : (0, \infty) \to \mathbb{R}$ are two strictly increasing, differentiable functions satisfying

$$f(g(x)) = x \quad \forall\, x \in (0, \infty)\,, \qquad g(f(x)) = x \quad \forall\, x \in \mathbb{R}\,,$$

and $f(a+b) = f(a)f(b)$ for all $a, b \in \mathbb{R}$. Show that $xf'(0)g'(x) = 1$ for all $x \in \mathbb{R}$.

Problem 6. (10%) Suppose that x and y satisfy the relation $y \sin x = x \sin y$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(\pi, 0)$ using the implicit differentiation.

Problem 7. Let $f : [-\pi, \pi] \to \mathbb{R}$ be given by $f(x) = -x^2 \cos x + 4x \sin x + 6 \cos x$.

(1) (5%) Find the inflection points of the graph of f.

(2) (5%) Use the first derivative test to find all the relative extrema of f'.

(2) (10%) Show that $|f(x) - f(y)| \leqslant 2\pi|x - y|$ for all $x, y \in [-\pi, \pi]$.