Calculus II Midterm 2

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Problem 1. Parametrize the curve

$$\vec{\mathbf{r}}(t) = (e^{2t}\cos 2t, 2, e^{2t}\sin 2t) = e^{2t}\cos 2t\,\vec{\mathbf{i}} + 2\,\vec{\mathbf{j}} + e^{2t}\sin 2t\,\vec{\mathbf{k}}$$

with respect to the arc-length measured from the point where $t = 0$ in the direction of increasing t, by completing the following:

1. (5%) Compute $\vec{\mathbf{r}}'(t)$.
2. (10%) Compute the arc-length function $s(t)$.
3. (10%) Invert the arc-length function and derive the arc-length parametrization of $\vec{\mathbf{r}}(t)$.

Sol:

1. By the definition of the derivative of a vector-valued function,

$$\begin{aligned}\vec{\mathbf{r}}'(t) &= (2e^{2t}(\cos 2t - \sin 2t), 0, 2e^{2t}(\sin 2t + \cos 2t)) \\ &= 2e^{2t}(\cos 2t - \sin 2t)\,\vec{\mathbf{i}} + 2e^{2t}(\sin 2t + \cos 2t)\,\vec{\mathbf{k}}\,.\end{aligned}$$

2. Since $|\vec{\mathbf{r}}'(t)| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + (\sin 2t + \cos 2t)^2} = 2\sqrt{2}e^{2t}$, the arc-length function $s(t)$ is

$$s(t) = \int_0^t |\vec{\mathbf{r}}'(t')|\,dt' = 2\sqrt{2}\int_0^t e^{2t'}\,dt' = \sqrt{2}e^{2t'}\Big|_{t'=0}^{t'=t} = \sqrt{2}(e^{2t} - 1).$$

3. Let $s = \sqrt{2}(e^{2t} - 1)$. Then $t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right) = \frac{1}{2}\ln\frac{s+\sqrt{2}}{\sqrt{2}}$. Therefore, the arc-length parametrization of $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}_1(s) = \vec{\mathbf{r}}\left(\frac{1}{2}\ln\frac{s+\sqrt{2}}{\sqrt{2}}\right) = \left(\frac{s+\sqrt{2}}{\sqrt{2}}\cos\ln\frac{s+\sqrt{2}}{\sqrt{2}}, 2, \frac{s+\sqrt{2}}{\sqrt{2}}\sin\ln\frac{s+\sqrt{2}}{\sqrt{2}}\right).$$

Problem 2. (15%) Let S be the graph of $f(x,y) = x^2 - 4xy - 2y^2 + 12x - 12y - 1$. What horizontal plane is tangent to the surface S and what is the point of tangency.

Sol: Since $f_x(x,y) = 2x - 4y + 12$ and $f_y(x,y) = -4x - 4y - 12$, $(x,y) = (-4,1)$ is the only critical point of f. At this point, the tangent plane is horizontal, and the tangent plane is $z = f(-4,1) = -31$. □

Problem 3. Let $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

1. (12%) Find the first partial derivative f_x and f_y for all $(x,y) \in \mathbb{R}^2$.

2. (10%) Show that f is differentiable at the origin $(0,0)$.

3. (8%) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

4. (5%) Let $\vec{\mathbf{u}} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Compute the directional derivative of f_x at $(0,0)$ in the direction $\vec{\mathbf{u}}$.

Proof.

1. If $(x,y) \neq (0,0)$, by the quotient rule we find that

$$\begin{aligned} f_x(x,y) &= \frac{\big[y(x^2-y^2)+2x^2y\big](x^2+y^2)-2x^2y(x^2-y^2)}{(x^2+y^2)^2} \\ &= \frac{y(x^2-y^2)(x^2+y^2)+2x^2y\big[(x^2+y^2)-(x^2-y^2)\big]}{(x^2+y^2)^2} \\ &= \frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}. \end{aligned}$$

On the other hand,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

Therefore,

$$f_x(x,y) = \begin{cases} \dfrac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

Since $f(x,y) = -f(y,x)$, $f_y(x,y) = -f_x(y,x)$; thus

$$f_y(x,y) = \begin{cases} \dfrac{-xy^4+x^5-4x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

2. In order to show that f is differentiable at $(0,0)$, we need to compute the limit

$$\lim_{(h,k)\to(0,0)}\frac{f(h,k)-\big[f(0,0)+f_x(0,0)h+f_y(0,0)\big]}{\sqrt{h^2+k^2}} = 0.$$

Nevertheless, letting $h = r\cos\theta$ and $k = r\sin\theta$, we find that

$$\begin{aligned} \lim_{(h,k)\to(0,0)}\frac{f(h,k)-\big[f(0,0)+f_x(0,0)h+f_y(0,0)\big]}{\sqrt{h^2+k^2}} &= \lim_{r\to 0}\frac{f(r\cos\theta, r\sin\theta)}{r} \\ = \lim_{r\to 0}\frac{\frac{r^4(\cos^3\theta\sin\theta-\cos\theta\sin^3\theta)}{r^2}}{r} &= \lim_{r\to 0} r(\cos^3\theta\sin\theta-\cos\theta\sin^3\theta) = 0. \end{aligned}$$

3. By definition,

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k^5/k^4}{k} = -1,$$

while on the other hand,

$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h^5/h^4}{h} = 1.$$

As a consequence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

4. By definition,

$$(D_{\vec{\mathbf{u}}} f_x)\Big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\Big) = \lim_{h\to 0} \frac{f_x\big(h\frac{\sqrt{2}}{2}, h\frac{\sqrt{2}}{2}\big) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{2h^5(\frac{\sqrt{2}}{2})^3}{h^5} = \frac{\sqrt{2}}{2}. \qquad \square$$

Problem 4. Let C be the plane curve $\Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, x^3 + 2y^3 = -3xy\Big\}$ (see figure 1 for reference).

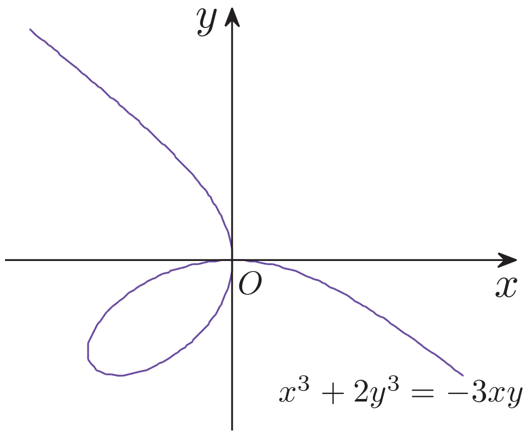

Figure 1

1. (10%) Find the line tangent to the curve C at the point $(2, -1)$.
2. (10%) Find the point at which the line tangent to the curve C is horizontal.
3. (10%) Find the point at which the line tangent to the curve C is vertical.

Sol: Let $F(x,y) = x^3 + 2y^3 + 3xy$. By the implicit differentiation,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+3y}{6y^2+3x} = -\frac{x^2+y}{2y^2+x}.$$

1. At $(x,y) = (2,-1)$, $\dfrac{dy}{dx} = -\dfrac{4-1}{2+2} = -\dfrac{3}{4}$ which is the slope of the tangent line. Therefore, the tangent line at $(2,-1)$ is
$$y = -\frac{3}{4}(x-2) - 1 = -\frac{3}{4}x + \frac{1}{2}.$$

2. If the tangent line at (a,b) is horizontal, then $a^2 + b = 0$. Moreover, (a,b) being on the curve C suggests that $a^3 + 2b^3 + 3ab = 0$. Therefore, $a^3 - 2a^6 - 3a^3 = 0$ which implies that $a = 0$ or $a = -1$. Since (according to the figure) there is no tangent line at the origin, $a = -1$ (and thus $b = -a^2 = -1$). Therefore, the point at which the line tangent to C is horizontal is $(-1,-1)$.

3. If the tangent line at (a,b) is vertical, then $2b^2 + a = 0$. Moreover, (a,b) being on the curve C suggests that $a^3 + 2b^3 + 3ab = 0$. Therefore, $-8b^6 + 2b^3 - 6b^3 = 0$ which implies that $b = 0$ or $b = -\dfrac{1}{\sqrt[3]{2}}$. Since (according to the figure) there is no tangent line at the origin, $b = -\dfrac{1}{\sqrt[3]{2}}$ (and thus $a = -2b^2 = -\sqrt[3]{2}$). Therefore, the point at which the line tangent to C is vertical is $\Big(-\sqrt[3]{2}, -\dfrac{1}{\sqrt[3]{2}}\Big)$.

Problem 5. Use the chain rule to compute the following partial derivatives.

1. (15%) Let $f(x,y) = \displaystyle\int_{x-y}^{x^2+y} \cos(t^2)\,dt$. Find f_{xy}.

2. (15%) Let $z = \sin^{-1}(x-y)$, and $x = s^2+t^2$ and $y = 1-2st$. Show that

$$\frac{\partial z}{\partial t} = -\frac{2}{\sqrt{2-(s+t)^2}} \qquad \text{if } -\sqrt{2} < s+t < 0.$$

Sol:

1. Let $F(x) = \displaystyle\int_a^x \cos(t^2)\,dt$. Then $f(x,y) = F(x^2+y) - F(x-y)$. By the chain rule,

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}F(x^2+y) - \frac{\partial}{\partial x}F(x-y) \\ &= F'(x^2+y)\cdot\frac{\partial(x^2+y)}{\partial x} - F'(x-y)\cdot\frac{\partial(x-y)}{\partial x} \\ &= 2x\cos(x^2+y)^2 - \cos(x-y)^2, \end{aligned}$$

where we use the fundamental theorem of calculus to obtain $F'(x) = \cos(x^2)$ to proceed the computation. Therefore,

$$\begin{aligned} f_{xy}(x,y) &= -2x\sin(x^2+y)^2\cdot\frac{\partial(x^2+y)^2}{\partial y} + \sin(x-y)^2\cdot\frac{\partial(x-y)^2}{\partial y} \\ &= -4x(x^2+y)\sin(x^2+y)^2 - 2(x-y)\sin(x-y)^2. \end{aligned}$$

2. Since $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$,

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$$

Therefore, for $s+t<0$,

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{2t}{\sqrt{1-(s^2+t^2-1+2st)^2}} - \frac{-2s}{\sqrt{1-(s^2+t^2-1+2st)^2}} \\ &= \frac{2(s+t)}{\sqrt{1^2-[(s+t)^2-1]^2}} = \frac{2(s+t)}{\sqrt{(s+t)^2[2-(s+t)^2]}} \\ &= \frac{2(s+t)}{|s+t|\sqrt{2-(s+t)^2}} = -\frac{2}{\sqrt{2-(s+t)^2}}. \end{aligned}$$

Problem 6. (15%) Show that the equation of the tangent plane to the elliptic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z+z_0}{c}. \tag{0.1}$$

Proof. Let $F(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z}{c}$. Since

$$(\nabla F)(x,y,z) = \Big(\frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{1}{c}\Big),$$

the normal direction at the point (x_0, y_0, z_0) is $(\nabla F)(x_0, y_0, z_0) = \Big(\dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, -\dfrac{1}{c}\Big)$; thus the tangent plane at (x_0, y_0, z_0) is

$$\Big(\frac{2x_0}{a^2}, \frac{2y_0}{b^2} - \frac{1}{c}\Big) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

or

$$\frac{2x_0 x}{a^2} + \frac{2y_0 y}{b^2} - \frac{z}{c} = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} - \frac{z_0}{c}. \tag{0.2}$$

Since (x_0, y_0, z_0) is on the elliptic paraboloid,

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0}{c} = 0;$$

thus (0.2) implies that the tangent plane at (x_0, y_0, z_0) is given by (0.1). □