Calculus II Final Exam

National Central University, Summer Session 2012, Sep. 4, 2012

Problem 1. (10%) Figure 1 shows the region of the integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dydzdx.$$

Rewrite this integral as an equivalent iterated integral in the order of $dxdzdy$.

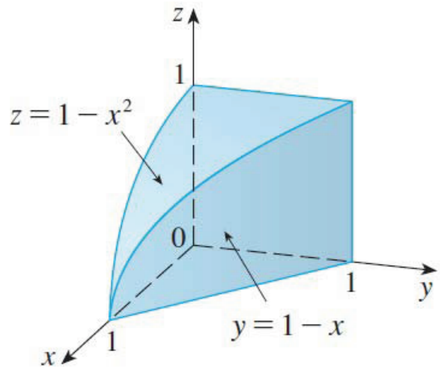
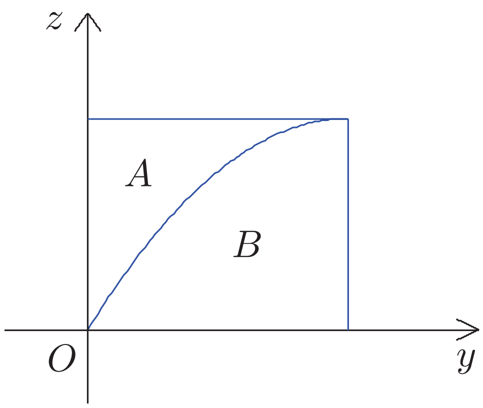


Figure 1

Sol: The projection of the solid onto the yz-plane is the unit square $[0,1]\times[0,1]$. Moreover, the intersection of the plane $y=1-x$ and the paraboloid $z=1-x^2$ is $z=1-(1-y)^2=2y-y^2$ which divides the unit square into two pieces. Let A be the piece adjacent to the z-axis and B be the piece adjacent to the y-axis.



In other words,

$$A \equiv \left\{(y,z)\ \middle|\ 0\le y\le 1, 2y-y^2\le z\le 1\right\},$$
$$B \equiv \left\{(y,z)\ \middle|\ 0\le y\le 1, 0\le z\le 2y-y^2\right\}.$$

Therefore, the integral can be also written as

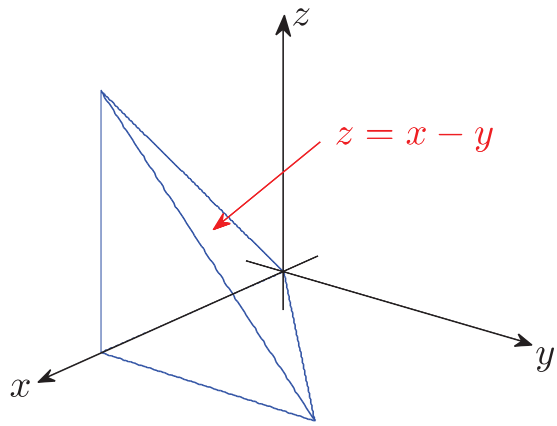
$$\iint_A \left[\int_0^{\sqrt{1-z}} f(x,y,z)dx\right]dzdy + \iint_B \left[\int_0^{1-y}\right]dxdzdy$$
$$= \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)dxdzdy + \int_0^1 \int_0^{2y-y^2} \int_0^{\sqrt{1-z}} f(x,y,z)dxdzdy. \qquad \square$$

Problem 2. (15%) Evaluate the triple integral

$$\iiint_{\mathrm{T}} xyz\, dV,$$

where T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(1,0,1)$.

Sol: First of all, the plane passing through $(0,0,0)$, $(1,1,0)$ and $(1,0,1)$ is $x-y-z=0$ or $z=x-y$.



Therefore, the tetrahedron T can be expressed as

$$\mathrm{T} = \Big\{(x,y,z)\ \Big|\ (x,y)\in D, 0\le z\le x-y\Big\},$$

where D is the triangular region on xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$. As a consequence,

$$\begin{aligned}\iiint_{\mathrm{T}} xyzdV &= \int_0^1\int_0^x\int_0^{x-y} xyzdzdydx = \int_0^1\int_0^x \frac{xy(x-y)^2}{2}dydx\\ &= \int_0^1\int_0^x \Big[-\frac{x(x-y)^3}{2}+\frac{x^2(x-y)^2}{2}\Big]dydx\\ &= \int_0^1 \Big[\frac{x(x-y)^4}{8}-\frac{x^2(x-y)^3}{6}\Big]\Big|_{y=0}^{y=x}dx\\ &= \int_0^1 \Big[-\frac{x^5}{8}+\frac{x^5}{6}\Big]dx = \frac{x^6}{144}\Big|_{x=0}^{x=1} = \frac{1}{144}.\end{aligned}$$

□

Problem 3. Find the volume of the solid that lies between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$ using

1. (15%) the cylindrical coordinate, and
2. (15%) the spherical coordinate.

Sol: First we find the intersection of the paraboloid and the sphere. If (x,y,z) is on the intersection, then $z+z^2=2$ which implies $z=1$ or $z=-2$ which is impossible. Therefore, the paraboloid and the sphere intersection at $x^2+y^2=1$.

1. Using the cylindrical coordinates, the paraboloid can be expressed as $z = r^2$ and the upper half sphere can be expressed as $z = \sqrt{2-r^2}$. Therefore, the required volume is

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r dz dr d\theta = \int_0^{2\pi}\int_0^1 \Big[r\sqrt{2-r^2} - r^3\Big] dr d\theta = 2\pi\Big[-\frac{1}{3}(2-r^2)^{\frac{3}{2}} - \frac{1}{4}r^4\Big]\Big|_{r=0}^{r=1}$$
$$= 2\pi\Big[\Big(-\frac{1}{3}-\frac{1}{4}\Big) - \Big(-\frac{2\sqrt{2}}{3}\Big)\Big] = \frac{2\pi}{3}\Big(2\sqrt{2}-\frac{7}{4}\Big).$$

2. Using the spherical coordinates, the paraboloid can be expressed as $\rho = \dfrac{\cos\varphi}{\sin^2\varphi}$, and the sphere can be expressed as $\rho = \sqrt{2}$. Therefore, the required volume is

$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho^2\sin\varphi d\rho d\theta d\varphi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{\cos\varphi}{\sin^2\varphi}} \rho^2\sin\varphi d\rho d\theta d\varphi$$
$$= \frac{2\sqrt{2}}{3}\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\sin\varphi d\varphi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{\cos^3\varphi}{3\sin^5\varphi} d\theta d\varphi$$
$$= \frac{2\sqrt{2}}{3}\times 2\pi\times\Big[-\cos\varphi\Big]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{4}} + \frac{2\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{(1-\sin^2\varphi)\cos\varphi}{\sin^5\varphi}d\varphi$$
$$= \frac{2\pi}{3}(2\sqrt{2}-2) + \frac{2\pi}{3}\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1-u^2}{u^5}du \qquad \text{(by letting } u = \sin\varphi)$$
$$= \frac{2\pi}{3}(2\sqrt{2}-2) + \frac{2\pi}{3}\Big[-\frac{1}{4}u^{-4} + \frac{1}{2}u^{-2}\Big]\Big|_{u=\frac{1}{\sqrt{2}}}^{u=1}$$
$$= \frac{2\pi}{3}(2\sqrt{2}-2) + \frac{2\pi}{3}\Big[\Big(-\frac{1}{4}+\frac{1}{2}\Big) - \Big(-\frac{4}{4}+\frac{2}{2}\Big)\Big]$$
$$= \frac{2\pi}{3}\Big(2\sqrt{2}-2+\frac{1}{4}\Big) = \frac{2\pi}{3}\Big(2\sqrt{2}-\frac{7}{4}\Big).$$

where we note that the first integral is the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the sphere, while the second integral is the volume of the solid below the cone and above the paraboloid. □

Problem 4. Let R be the region bounded by $y = 3x$, $y = \sqrt{3}$ and the hyperbola $xy = 3$. Find the double integral $\iint_R xy\, dA$ by

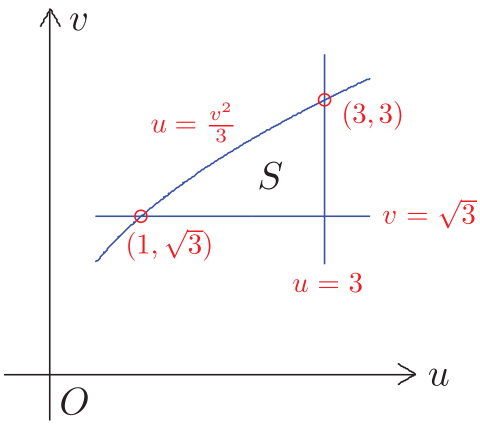
1. (10%) Plot the region S on the uv-plane which corresponds R on the xy-plane.
2. (15%) Use the change of variables $x = \dfrac{u}{v}$ and $y = v$ and the change of variable formula to compute the double integral.

Sol:

1. Since $x = \dfrac{u}{v}$ and $y = v$, the curve on the uv-plane corresponding to $y = 3x$ is

$$v = \frac{3u}{v} \quad \text{or} \quad u = \frac{v^2}{3},$$

while the hyperbola $xy = 3$ on the xy-plane corresponds to $u = 3$ on the uv-plane. Moreover, the curve corresponding to $y = \sqrt{3}$ on the xy-plane is $v = \sqrt{3}$ on uv-plane. Therefore,



2. The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

Therefore,

$$\iint_R xydA = \iint_S \frac{u}{|v|} dudv = \int_{\sqrt{3}}^{3} \int_{\frac{v^2}{3}}^{3} \frac{u}{v} dudv = \int_{\sqrt{3}}^{3} \Big[\frac{9}{2v} - \frac{v^3}{18}\Big] dv$$
$$= \Big[\frac{9}{2} \ln v - \frac{v^4}{72}\Big]\Big|_{v=\sqrt{3}}^{v=3} = \Big[\Big(\frac{9}{2} \ln 3 - \frac{81}{72}\Big) - \Big(\frac{9}{4} \ln 3 - \frac{9}{72}\Big)\Big] = \frac{9}{4} \ln 3 - 1$$

or

$$\iint_R xydA = \iint_S \frac{u}{|v|} dudv = \int_1^3 \int_{\sqrt{3}}^{\sqrt{3u}} \frac{u}{v} dvdu = \int_1^3 \Big[\frac{u}{2} \ln(3u) - \frac{u}{2} \ln 3\Big] du$$
$$= \frac{1}{2} \int_1^3 u \ln u du = \frac{1}{2}\Big[\frac{u^2 \ln u}{2}\Big|_{u=1}^{u=3} - \int_1^3 \frac{u}{2} du\Big] = \frac{1}{2}\Big[\frac{9 \ln 3}{2} - \frac{8}{4}\Big] = \frac{9}{4} \ln 3 - 1.$$ □

Problem 5. Let $\vec{\mathbf{F}}(x, y) = (ye^x + \sin y)\,\vec{\mathbf{i}} + (e^x + x \cos y)\,\vec{\mathbf{j}}$.

1. (10%) Show that $\vec{\mathbf{F}}$ is a conservative vector field; that is, find a scalar potential φ such that $\vec{\mathbf{F}} = \nabla\varphi$.

2. (10%) Let C be a curve given by $\vec{\mathbf{r}}(t) = (\cos t, \sin t)$ with $0 \le t \le \pi$. Compute the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ by the fundamental theorem of line integrals.

Sol:

1. Suppose that $\vec{\mathbf{F}} = \nabla\phi$. Then $\phi_x = ye^x + \sin y$ and $\phi_y = e^x + x \cos y$. Therefore, $\phi(x, y) = ye^x + x \sin y + C_1(y)$ and $\phi(x, y) = ye^x + x \sin y + C_2(x)$. This implies that $C_1(y) = C_2(x)$; thus $C_1 = C_2 = \text{const}$. Therefore, if

$$\phi(x, y) = ye^x + x \sin y,$$

then $\vec{\mathbf{F}} = \nabla\phi$ which implies that $\vec{\mathbf{F}}$ is conservative.

2. Since $\vec{\mathbf{F}} = \nabla\phi$ is conservative, by the fundamental theorem of line integrals,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \phi(\vec{\mathbf{r}}(\pi)) - \phi(\vec{\mathbf{r}}(0)) = \phi(-1,0) - \phi(1,0) = 0. \qquad \square$$

Problem 6. Let $T \equiv \left\{(u,v) \;\middle|\; 0 \le v \le \frac{\pi}{2}, v \le u \le \pi - v\right\}$ be a triangular region on uv-plane, and $\vec{\mathbf{r}}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$, $(u,v) \in T$, be a parametrization of a part of a surface $\mathcal{S}$ on the sphere.

1. (10%) Compute $|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|$.
2. (10%) Compute the surface integral $\displaystyle\int_{\mathcal{S}} y\, dS$.

Sol: First we compute $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$ as follows:

$$\begin{aligned}\vec{\mathbf{r}}_u(u,v) &= (-\sin u \sin v, \cos u \sin v, 0),\\ \vec{\mathbf{r}}_v(u,v) &= (\cos u \cos v, \sin u \cos v, -\sin v).\end{aligned}$$

1. Therefore,

$$\begin{aligned}\vec{\mathbf{r}}_u(u,v) \times \vec{\mathbf{r}}_v(u,v) &= (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v)\\ &= (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v);\end{aligned}$$

   thus

$$\begin{aligned}|\vec{\mathbf{r}}_u(u,v) \times \vec{\mathbf{r}}_v(u,v)| &= \sqrt{\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \sin^2 v \cos^2 v}\\ &= \sqrt{\sin^4 v + \sin^2 v \cos^2 v)} = \sqrt{\sin^2 v} = |\sin v| = \sin v,\end{aligned}$$

   where the last equality is based on $0 \le v \le \frac{\pi}{2}$ which makes $\sin v$ non-negative.

2. By definition,

$$\begin{aligned}\int_{\mathcal{S}} y\, dS &= \int_0^{\frac{\pi}{2}} \int_v^{\pi - v} \sin u \sin v \cdot \sin v\, du dv = \int_0^{\frac{\pi}{2}} \int_v^{\pi - v} \sin u \sin^2 v du dv\\ &= -\int_0^{\frac{\pi}{2}} \left[\cos u \Big|_{u=v}^{u=\pi - v}\right] \sin^2 v dv = 2\int_0^{\frac{\pi}{2}} \cos v \sin^2 v dv = \frac{2}{3}\sin^3 v \Big|_{v=0}^{v=\frac{\pi}{2}} = \frac{2}{3}. \qquad \square\end{aligned}$$