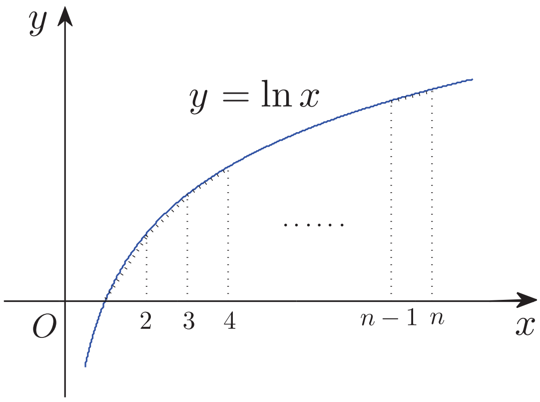
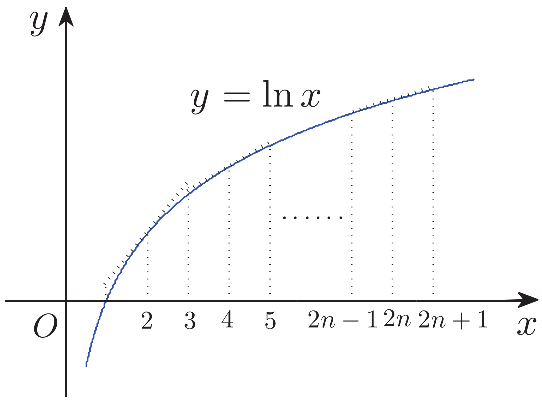
Problem: Complete the following.

1. By looking at the following two pictures, show that



(a) Under-estimate of $\int_1^n \ln x dx$



(b) Over-estimate of $\int_1^{2n+1} \ln x dx$

$$\ln n! - \frac{1}{2}\ln n \le n\ln n - n + 1\,. \tag{1}$$

and

$$\Big(n+\frac{1}{2}\Big)\ln\Big(n+\frac{1}{2}\Big) + \frac{1}{2}\ln 2 - n \le \ln n!. \tag{2}$$

2. Suppose that we know that

$$\left(1+\frac{1}{2n}\right)^{n+0.5} \ge \sqrt{e}$$

By rearranging the inequalities (1) and (2), show that

$$\sqrt{2e}\,n^{n+0.5}e^{-n} \le n! \le e n^{n+0.5}e^{-n}.$$

3. For $n \ge 2$, we have the recursive formula

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

Show that for $k \in \mathbb{N}$,

$$I_{2k} \equiv \int_0^{\frac{\pi}{2}} \sin^{2k} x dx = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{(2k)!}{(2^k k!)^2}\frac{\pi}{2}$$

and

$$I_{2k+1} \equiv \int_0^{\frac{\pi}{2}} \sin^{2k+1} x dx = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\frac{2}{3} = \frac{(2^k k!)^2}{(2k+1)!}.$$

Note that you need to show the second equality in both identities.

4. Show that $\lim_{k\to\infty}\frac{I_{2k+1}}{I_{2k}} = 1$ by observing that

$$\frac{I_{2k+2}}{I_{2k}} \le \frac{I_{2k+1}}{I_{2k}} \le 1.$$

Moreover, also show that

$$\lim_{n\to\infty}\frac{n!}{n^{n+0.5}e^{-n}} = \sqrt{2\pi}. \tag{3}$$

In other words, $n!$ grows like $\sqrt{2\pi n}\,n^n e^{-n}$ as $n \to \infty$. (3) is called the Stirling's formula.