Calculus II Midterm 2

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Problem 1. Let a function f of two variables be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

and $\vec{u} = (\cos\theta, \sin\theta)$ be a unit vector.

1. (5%) Compute $D_{\vec{u}} f(0,0)$, the directional derivative of f in the direction $\vec{u}$ at the point $(0,0)$.
2. (5%) Compute the gradient of f at the point $(0,0)$.
3. (3%) Is $D_{\vec{u}} f(0,0)$ the same as $\nabla f(0,0) \cdot \vec{u}$?
4. (5%) Does the existence of the directional derivatives of a function in every direction imply the continuity of this function? Why?

Sol:

1. By definition,

$$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta)}{t}$$

If $\theta \neq \dfrac{\pi}{2}$ or $\theta \neq \dfrac{3\pi}{2}$ (that is, $\cos\theta \neq 0$),

$$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{t^3 \cos\theta \sin^2\theta}{t(t^2\cos^2\theta + t^4 \sin^4\theta)} = \frac{\sin^2\theta}{\cos\theta}\,.$$

If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$ (that is, $\cos\theta = 0$), $D_{\vec{u}} f(0,0) = 0\,.$

2. $f_x(0,0) = D_{(1,0)} f(0,0) = \dfrac{\sin^2 0}{\cos 0} = 0$. $f_y(0,0) = D_{(0,1)} f(0,0) = 0$. Therefore,

$$\nabla f(0,0) = (0,0)\,.$$

3. Take $\theta = \dfrac{\pi}{4}$. Then $D_{\vec{u}} f(0,0) = \dfrac{\sqrt{2}}{2}$, but $\nabla f(0,0) \cdot \vec{u} = 0$. Therefore, in general $D_{\vec{u}} f(0,0)$ is **not** the same as $\nabla f(0,0) \cdot \vec{u}$.

4. First we show that f is not continuous at $(0,0)$. In fact, along the curve $x = y^2$, f takes the value $\dfrac{1}{2}$. Therefore, f cannot be continuous at $(0,0)$. However, from (a) we know that the directional derivative of f exists along every direction. Therefore, the existence of the directional derivatives of a function in every direction does **not** imply the continuity of the function.

Problem 2. (20%) Let $w = f(x, y, z)$, and (ρ, θ, ϕ) be the spherical coordinate; that is, $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$. Show that

$$\|\nabla f\|^2 = \left(\frac{\partial w}{\partial \rho}\right)^2 + \frac{1}{\rho^2\sin^2\phi}\left(\frac{\partial w}{\partial \theta}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial w}{\partial \phi}\right)^2.$$

Sol: By the chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} = \cos\theta\sin\phi\frac{\partial w}{\partial x} + \sin\theta\sin\phi\frac{\partial w}{\partial y} + \cos\phi\frac{\partial w}{\partial z}\,, \\
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = -\rho\sin\theta\sin\phi\frac{\partial w}{\partial x} + \rho\cos\theta\sin\phi\frac{\partial w}{\partial y}\,, \\
\frac{\partial w}{\partial \phi} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \phi} = \rho\cos\theta\cos\phi\frac{\partial w}{\partial x} + \rho\sin\theta\cos\phi\frac{\partial w}{\partial y} - \rho\sin\phi\frac{\partial w}{\partial z}\,.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\left(\frac{\partial w}{\partial \rho}\right)^2 + \frac{1}{\rho^2\sin^2\phi}\left(\frac{\partial w}{\partial \theta}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial w}{\partial \phi}\right)^2 \\
&\quad= \cos^2\theta\sin^2\phi\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\sin^2\phi\left(\frac{\partial w}{\partial y}\right)^2 + \cos^2\phi\left(\frac{\partial w}{\partial z}\right)^2 + 2\sin\theta\cos\theta\sin^2\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
&\qquad +2\sin\theta\sin\phi\cos\phi\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} + 2\cos\theta\sin\phi\cos\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} + \frac{1}{\rho^2\sin^2\phi}\Big[\rho^2\sin^2\theta\sin^2\phi\left(\frac{\partial w}{\partial x}\right)^2 \\
&\qquad +\rho^2\cos^2\theta\sin^2\phi\left(\frac{\partial w}{\partial y}\right)^2 - 2\rho^2\sin\theta\cos\theta\sin^2\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big] + \frac{1}{\rho^2}\Big[\rho^2\cos^2\theta\cos^2\phi\left(\frac{\partial w}{\partial x}\right)^2 \\
&\qquad +\rho^2\sin^2\theta\cos^2\phi\left(\frac{\partial w}{\partial y}\right)^2 + \rho^2\sin^2\phi\left(\frac{\partial w}{\partial z}\right)^2 + 2\rho^2\sin\theta\cos\theta\cos^2\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
&\qquad -2\rho^2\sin\theta\sin\phi\cos\phi\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} - 2\rho^2\cos\theta\sin\phi\cos\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\Big] \\
&\quad= \cos^2\theta\sin^2\phi\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\sin^2\phi\left(\frac{\partial w}{\partial y}\right)^2 + \cos^2\phi\left(\frac{\partial w}{\partial z}\right)^2 + 2\sin\theta\cos\theta\sin^2\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
&\qquad +2\sin\theta\sin\phi\cos\phi\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} + 2\cos\theta\sin\phi\cos\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial z} + \sin^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \cos^2\left(\frac{\partial w}{\partial y}\right)^2 \\
&\qquad -2\sin\theta\cos\theta\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \cos^2\theta\cos^2\phi\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\cos^2\phi\left(\frac{\partial w}{\partial y}\right)^2 + \sin^2\phi\left(\frac{\partial w}{\partial z}\right)^2 \\
&\qquad +2\sin\theta\cos\theta\cos^2\phi\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2\sin\theta\sin\phi\cos\phi\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} - 2\cos\sin\phi\cos\phi\frac{\partial w}{\partial x}\partial w\partial z \\
&\quad= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.
\end{aligned}$$

Problem 3. Let $F(x, y, z) = \sin(xy) + e^{yz} + \ln(xz) - 1$.

1. (10%) Find the tangent plane P for the zero level surface of F at the point $(1, 0, 1)$.

2. (5%) Find the inclination angle of P; that is , find the angle between the plan P and the xy-plane.

Sol:

1. $\nabla F = \left(y\cos(xy) + \frac{1}{x}, x\cos(xy) + ze^{yz}, ye^{yz} + \frac{1}{z}\right)$. Therefore, the tangent plane at $(1,0,1)$ is
$$\nabla F(1,0,1)\cdot(x-1,y,z-1) = 0 \qquad \text{or} \qquad x + 2y + z = 2\,.$$

2. The normal direction of plane P is $(1,2,1)$. Therefore, if the inclination angle is θ, then
$$\cos\theta = \frac{(1,2,1)\cdot(0,0,1)}{\|(1,2,1)\|} = \frac{1}{\sqrt{6}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{6}}\,.$$

Problem 4. (12%) Complete the following.

1. (2%) Read the following statement, and judge if the statement is **T**rue or **F**alse.

   > if a function of one variable is continuous on an interval and has only one critical point, then a local maximum has to be an absolute maximum.

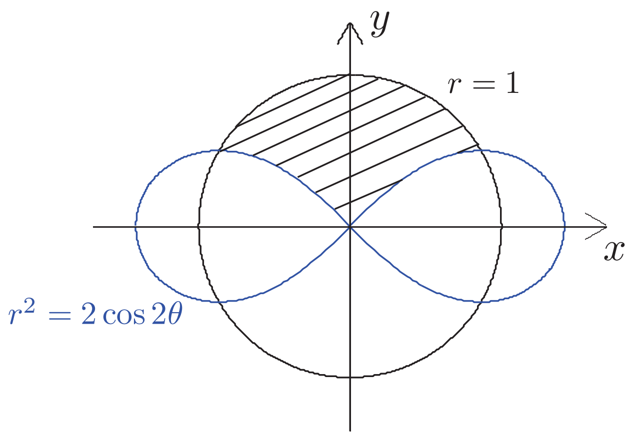
2. (10%) Show that the function
$$f(x,y) = 3xe^y - x^3 - e^{3y}$$
has exactly one critical point, and that f has a local maximum there that is no absolute maximum.

Sol:

1. True.

2. $(\nabla f)(x,y) = (3e^y - 3x^2, 3xe^y - 3e^{3y})$. Then $(\nabla f)(x,y) = (0,0)$ if and only if $(x,y) = (1,0)$. This implies that there is only one critical point of the function f. Moreover,
$$(D^2 f)(x,y) = \begin{bmatrix} -6x & 3e^y \\ 3e^y & 3xe^y - 9e^{3y} \end{bmatrix};$$
thus $(D^2 f)(1,0) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$ which is negative definite. Therefore, f has a local maximum at $(1,0)$.

Problem 5. (15%) Let R be the region bounded the circle $r = 1$ and outside the lemniscate $r^2 = 2\cos 2\theta$, and is located on the upper half plane (see the shaded region in the graph).



Find the maximum value of the function $f(x, y) = y^4 + 2x^2y^2 + 2y^2$ with (x, y) located in the region R.

Sol: The maximum may occur in the interior of R or on the boundary or R.

1. If the maximum occurs in the interior of R. Then $\nabla f = (0, 0)$ at that point. However, $(\nabla f)(x, y) = (4xy^2, 4y^3 + 4x^2y + 4y) = (0, 0)$ if and only if $(x, y) = (x, 0)$ for all x. In such case, $f(x, y) = 0$.

2. If the maximum occurs on the boundary $r = 1$. Let

$$F(x, y, \lambda) = y^4 + 2x^2y^2 + 2y^2 + \lambda(x^2 + y^2 - 1).$$

   Then $(F_x, F_y, F_\lambda) = (4xy^2 + 2\lambda x, 4y^3 + 4x^2y + 4y + 2\lambda y, x^2 + y^2 - 1)$, and $(F_x, F_y, F_\lambda) = (0, 0, 0)$ if and only if

$$\begin{aligned} 2x(2y^2 + \lambda) &= 0, \\ 2y(2x^2 + 2y^2 + 2 + \lambda) &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

   Therefore, $(x, y, \lambda) = (0, \pm 1, -4)$ or $(x, y, \lambda) = (\pm 1, 0, 0)$. Only the point $(x, y) = (0, 1)$ is on the boundary of R, and $f(0, 1) = 3$.

3. If the maximum occurs on the boundary of lemniscate. If (x, y) is on the lemniscate, then

$$x^4 + 2x^2y^2 + y^4 = 2(x^2 - y^2) \quad \text{or} \quad y^4 + 2x^2y^2 + 2y^2 = 2x^2 - x^4.$$

   In other words, we need to maximize the function $g(x) = 2x^2 - x^4$ in the interval $(-\sqrt{3}/2, \sqrt{3}/2)$. The only critical point of g in this interval is $x = 0$. At the endpoints of the interval, $g = 15/16$.

4. Comparing all possible maximum, we find that the maximum of f over the region R is 3.

Problem 6. Let $P_1 = (1, 0)$, $P_2 = (1, 1)$, $P_3 = (0, -1)$ and $P_4 = (1, -1)$ be four points on the plane.

1. (10%) Find a straight line L so that the sum of the squared distance

$$S = \sum_{i=1}^{4} \text{dist}(P_i, L)^2$$

   is smallest, where $\text{dist}(P, L)$ denotes the distance from a point P to line L.

2. (10%) Fine the least square regression line.

Sol:

1. Assume that the line L is $\cos\theta x + \sin\theta y + k = 0$ for some θ and k. Then

$$S(\theta, k) = \sum_{i=1}^{4} (x_i \cos\theta + y_i \sin\theta + k)^2 ,$$

where $P_i = (x_i, y_i)$. We then have $\sum_{i=1}^{4} x_i^2 = 3$, $\sum_{i=1}^{4} y_i^2 = 3$, $\sum_{i=1}^{4} x_i y_i = 0$, $\sum_{i=1}^{4} x_i = 3$, $\sum_{i=1}^{4} y_i = -1$, and

$$\begin{aligned} S_\theta(\theta,k) &= 2\sum_{i=1}^{4}(x_i\cos\theta + y_i\sin\theta + k)(-x_i\sin\theta + y_i\cos\theta) \\ &= 2\sin\theta\cos\theta\sum_{i=1}^{4}(y_i^2 - x_i^2) + 2(\cos^2\theta - \sin^2\theta)\sum_{i=1}^{4}x_i y_i \\ &\quad + 2k\cos\theta\sum_{i=1}^{4} y_i - 2k\sin\theta\sum_{i=1}^{4} x_i \\ &= -2k\cos\theta - 6k\sin\theta\,. \\ S_k(\theta,k) &= 2\sum_{i=1}^{4}(x_i\cos\theta + y_i\sin\theta + k) = 6\cos\theta - 2\sin\theta + 8k\,. \end{aligned}$$

The critical point of S satisfies

$$S_\theta(\theta,k) = 0 \Rightarrow k\cos\theta + 3k\sin\theta = 0 \Rightarrow k = 0 \text{ or } \cos\theta = -3\sin\theta\,.$$

We also have

$$\begin{aligned} S_{\theta\theta}(\theta,k) &= 2k\sin\theta - 6k\cos\theta\,, \\ S_{\theta k}(\theta,k) &= -6\sin\theta - 2\cos\theta\,, \\ S_{kk}(\theta,k) &= 8\,. \end{aligned}$$

(i) If $k = 0$, then $S_k(\theta,k) = 0$ implies $3\cos\theta = \sin\theta$. On the other hand, $S_{\theta\theta}S_{kk} - S_{\theta k}^2 = -S_{\theta k}^2 \le 0$. In order to attain extrema, $S_{\theta k}^2$ has to be zero, and this implies $3\sin\theta = -\cos\theta$. Therefore, $\sin\theta = \cos\theta = 0$ which is impossible.

(ii) If $\cos\theta = -3\sin\theta$, then $S_k(\theta,k) = 0$ implies $k = \frac{5}{2}\sin\theta$. Furthermore, for this to hold we have

$$\cos\theta = \pm\frac{3}{\sqrt{10}}\,, \quad \sin\theta = \mp\frac{1}{\sqrt{10}}\,, \quad k = \mp\frac{5}{2\sqrt{10}}\,;$$

hence $S_{\theta\theta}S_{kk} - S_{\theta k}^2 = S_{\theta\theta}S_{kk} > 0$. So S attains a relative minimum for this kind of critical points. It is in fact an absolute minimum. In this case, $L : 3x - y = 2.5$.

2. Suppose the least square regression line is $y = ax + b$. Then

$$\begin{aligned} a &= \frac{4\sum_{i=1}^{4} x_i y_i - \sum_{i=1}^{4} x_i \sum_{i=1}^{4} y_i}{4\sum_{i=1}^{4} x_i^2 - (\sum_{i=1}^{4} x_i)^2} = 1\,, \\ b &= \frac{1}{4}\Big(\sum_{i=1}^{4} y_i - a\sum_{i=1}^{4} x_i\Big) = -1\,. \end{aligned}$$