# Calculus II Midterm 1

National Central University, Spring 2012, Apr. 5, 2012

**Problem 1.** (10%) Evaluate the definite integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2+\sin 2x}dx$.

*Sol.* Let $u = 2x$. Then $du = 2dx$; thus

$$\int_0^{\frac{\pi}{4}} \frac{1}{2+\sin 2x}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin u}du.$$

By the change of variable $t = \tan\dfrac{u}{2}$, we find that

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{2+\sin 2x}dx &= \frac{1}{2}\int_0^1 \frac{1}{2+\frac{2t}{1+t^2}}\frac{2dt}{1+t^2} = \frac{1}{2}\int_0^1 \frac{1}{t^2+t+1}dt \\ &= \frac{1}{2}\int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}\Big|_{t=0}^{t=1} \\ &= \frac{1}{\sqrt{3}}\Big[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\Big] = \frac{1}{\sqrt{3}}\Big[\frac{\pi}{3}-\frac{\pi}{6}\Big] = \frac{\pi}{6\sqrt{3}}.\end{aligned}$$

□

**Problem 2.** (10%) Find all $\alpha \in \mathbb{R}$ so that the improper integral $\displaystyle\int_{e^2}^{\infty} \frac{1}{x\big[\ln\ln(1+x)\big]^{\alpha}}dx$ is convergent.

*Sol.* Let $e^y = 1 + x$. Then

$$\int_{e^2}^{\infty} \frac{1}{x\big[\ln\ln(1+x)\big]^{\alpha}}dx = \int_{\ln(1+e^2)}^{\infty} \frac{e^y dy}{(e^y-1)(\ln y)^{\alpha}} \geq \int_{\ln(1+e^2)}^{\infty} \frac{dy}{(\ln y)^{\alpha}} = \int_{\ln\ln(1+e^2)}^{\infty} \frac{e^u du}{u^{\alpha}},$$

where we use the change of variable $u = \ln y$ to conclude the last equality. Since $\lim_{u\to\infty} e^u u^{-\alpha} = \infty$ for all $\alpha > 0$, the improper integral is divergent (to $\infty$) for all $\alpha > 0$.

**Problem 3.** (10%) Let $f^{(k)}$ denote $\dfrac{d^k f}{dx^k}$, the $k$-th derivative of $f$, and $f^{(0)} \equiv f$. Suppose that $f^{(k)} : [-1, 1] \to \mathbb{R}$ is continuous for all $k \in \mathbb{N} \cup \{0\}$. Show that

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \cdots + \frac{h^n}{n!}f^{(n)}(0) + (-1)^n\int_0^h \frac{(x-h)^n}{n!}f^{(n+1)}(x)\,dx \tag{1}$$

by the integration by parts formula and induction.

*Proof.* By the fundamental theorem of Calculus and integration by parts,

$$\begin{aligned}f(h) &= f(0) + \int_0^h f'(x)dx = f(0) + (x-h)f'(x)\Big|_{x=0}^{x=h} - \int_0^h (x-h)f''(x)dx \\ &= f(0) + hf'(0) - \int_0^h (x-h)f''(x)dx.\end{aligned}$$

This prove the case $n = 1$.

Integrating by parts again suggests that

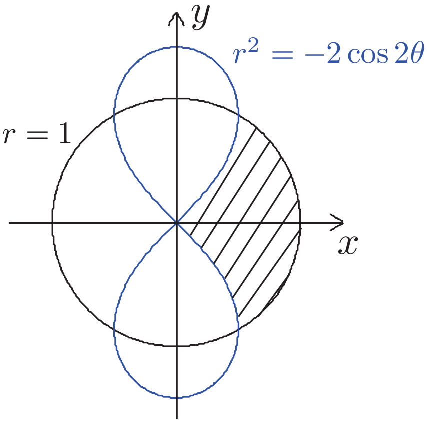
$$\int_0^h \frac{(x-h)^N}{N!} f^{(N+1)}(x)dx = \frac{(x-h)^{N+1}}{(N+1)!} f^{(N+1)}(x)\Big|_{x=0}^{x=h} - \int_0^h \frac{(x-h)^{N+1}}{(N+1)!} f^{(N+2)}(x)dx$$
$$= \frac{(-1)^{N+2}h^{N+1}}{(N+1)!} f^{(N+1)}(0) - \int_0^h \frac{(x-h)^{N+1}}{(N+1)!} f^{(N+2)}(x)dx.$$

Now suppose that (**??**) holds for $n = N$. Then the identity above implies that

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \cdots + \frac{h^N}{N!} f^{(N)}(0) + (-1)^N \int_0^h \frac{(x-h)^N}{N!} f^{(N+1)}(x)\,dx$$
$$= f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \cdots + \frac{h^N}{N!} f^{(N)}(0) + \frac{h^{N+1}}{(N+1)!} f^{(N+1)}(0)$$
$$+ (-1)^{N+1} \int_0^h \frac{(x-h)^{N+1}}{(N+1)!} f^{(N+2)}(x)dx.$$

This implies that (1) also holds for $n = N+1$. Therefore, (**??**) holds for all $n \in \mathbb{N} \cup \{0\}$ by induction.
□

**Problem 4.** Let $R$ be the region bounded by the circle $r = 1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).




1. (8%) Find the points of intersection of the circle $r = 1$ and the lemniscate $r^2 = -2\cos 2\theta$.

2. (7%) Show that the straight line $x = \dfrac{1}{2}$ is tangent to the lemniscate at the points of intersection on the right half plane.

3. (10%) Find the area of $R$.

4. Find the volume of the solid of revolution obtained by rotating $R$ about the $x$-axis by complete the following:

   (a) (5%) Suppose that $(x, y)$ is on the lemniscate. Then $(x, y)$ satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{2}$$

   for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) (3%) Solving (2), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1-4x^2}$ for some constants $c_1$, $c_2$ and $c_3$. Then the volume of interests can be computed by

$$I = \pi \int_0^{\frac{1}{2}} c(x)dx + \pi \int_{\frac{1}{2}}^1 d(x)dx.$$

Compute $\displaystyle\int_{\frac{1}{2}}^1 \Big[d(x) - (1-x^2)\Big]dx.$

(c) (12%) Evaluate $I$ by first computing the integral $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-4x^2}dx$, and then find $I$.

5. (10%) Find the area of the surface of revolution obtained by rotating the boundary of $R$ about the $x$-axis.

*Sol.*

1. Let $2\cos 2\theta = -1$, then $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$; thus the points of intersection are

$$\Big(\frac{1}{2}, \frac{\sqrt{3}}{2}\Big), \Big(\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big), \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big), \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big).$$

2. On the lemniscate, $r = \pm\sqrt{-2\cos 2\theta}$; thus

$$\frac{dx}{d\theta}\Big|_{\theta=\frac{\pi}{3}} = \Big[r'(\theta)\cos\theta - r(\theta)\sin\theta\Big]\Big|_{\theta=\frac{\pi}{3}} = \sqrt{2}\Big[\frac{\sin 2\theta}{\sqrt{-\cos 2\theta}}\cos\theta - \sqrt{-\cos 2\theta}\sin\theta\Big]\Big|_{\theta=\frac{\pi}{3}} = 0.$$

Similar computation shows that $\frac{dx}{d\theta}\Big|_{\theta=\frac{2\pi}{3}} = 0$; thus $x = \frac{1}{2}$ is tangent to the lemniscate.

3. The area of the shaded region is

$$2 \times \frac{1}{2}\Big[\int_0^{\frac{\pi}{4}} 1^2 d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + 2\cos 2\theta)d\theta\Big] = \frac{\pi}{4} + \big(\theta + \sin 2\theta\big)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1.$$

4. If $(x, y)$ is on the lemniscate, then

$$x^2 + y^2 = -2\Big(2\frac{x^2}{x^2+y^2} - 1\Big) = \frac{2(y^2-x^2)}{x^2+y^2}$$

which implies that

$$y^4 + 2(x^2-1)y^2 + x^4 + 2x^2 = 0.$$

Therefore,

$$y^2 = -(x^2-1) + \sqrt{(x^2-1)^2 - (x^4+2x^2)} = 1 - x^2 + \sqrt{1-4x^2}.$$

Therefore, the volume of the solid of revolution obtained by rotating $R$ about the $y$-axis is

$$\begin{aligned}
&\pi \int_0^{\frac{1}{2}} \Big[1 - x^2 + \sqrt{1-4x^2}\Big] dx + \pi \int_{\frac{1}{2}}^1 (1-x^2) dx \\
&\quad = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \pi \int_0^1 (1-x^2) dx \\
&\quad = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \pi \Big(x - \frac{x^3}{3}\Big)\Big|_{x=0}^{x=1} = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \frac{2\pi}{3}.
\end{aligned}$$

On the other hand, the integral can be evaluated by making a change of variable $x = \dfrac{\sin\theta}{2}$:

$$\begin{aligned}
\int \sqrt{1-4x^2}\, dx &= \frac{1}{2} \int \cos^2\theta d\theta = \frac{1}{4} \int \big(1 + \cos 2\theta\big) d\theta \\
&= \frac{1}{4}\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{4}\big(\theta + \sin\theta\cos\theta\big) + C \\
&= \frac{1}{4}\big(\sin^{-1} 2x + 2x\sqrt{1-4x^2}\,\big) + C.
\end{aligned}$$

Therefore,

$$\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx = \frac{\pi}{8}$$

and the volume of the solid of revolution obtained by rotating $R$ about the $x$-axis is $\dfrac{2\pi}{3} + \dfrac{\pi^2}{8}$.

5. There are two parts of the surface: one from rotating the lemniscate and the other from rotating the sphere. The area obtained by rotating the part of the lemniscate is

$$\begin{aligned}
\int 2\pi|y|ds &= \int 2\pi|r\sin\theta|\sqrt{r'^2 + r^2} d\theta \\
&= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{-2\cos 2\theta}\sin\theta \sqrt{(\sqrt{-2\cos 2\theta}')^2 + (-\sqrt{2\cos 2\theta})^2 d\theta} \\
&= 4\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{-\cos 2\theta}\sin\theta \frac{1}{\sqrt{-\cos 2\theta}}\, d\theta \\
&= 4\pi(-\cos\theta)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = 2\pi(\sqrt{2}-1).
\end{aligned}$$

The part obtained by rotating the part of the sphere is

$$\int 2\pi|y|ds = 2\pi \int_0^{\frac{\pi}{3}} \sin\theta\sqrt{1'^2 + 1^2} d\theta = 2\pi(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{3}} = \pi.$$

The total area is then $(2\sqrt{2}-1)\pi$. □

**Problem 5.** (15%) Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \sin^{-1}\frac{t}{\sqrt{1+t^2}}\mathbf{i} + \tan^{-1} t\mathbf{j} + \cos^{-1}\frac{1}{\sqrt{1+t^2}}\mathbf{k}, \quad t \in [-1, 1],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

*Sol.* By

$$\frac{d}{dt}\sin^{-1}\frac{t}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1-\frac{t^2}{1+t^2}}}\frac{\sqrt{1+t^2}-\frac{t^2}{\sqrt{1+t^2}}}{1+t^2} = \frac{1}{1+t^2},$$

$$\frac{d}{dt}\cos^{-1}\frac{1}{\sqrt{1+t^2}} = \frac{-1}{\sqrt{1-\frac{1}{1+t^2}}}\frac{-\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{1}{1+t^2},$$

$$\frac{d}{dt}\tan^{-1} t = \frac{1}{1+t^2},$$

we compute the arc-length function as

$$s(t) = \int_0^t \sqrt{\frac{1}{(1+t'^2)^2}+\frac{1}{(1+t'^2)^2}+\frac{1}{(1+t'^2)^2}}\,dt' = \sqrt{3}\int_0^t \frac{1}{1+t'^2}\,dt' = \sqrt{3}\tan^{-1} t.$$

Therefore, plugging in $t = \tan\frac{s}{\sqrt{3}}$, by

$$\sin^{-1}\frac{t}{\sqrt{1+t^2}} = \tan^{-1} t = \cos^{-1}\frac{1}{\sqrt{1+t^2}} = \frac{s}{\sqrt{3}},$$

we find that the required arc-length parametrization is

$$\mathbf{r}_1 = \mathbf{r}_1(s) = \frac{s}{\sqrt{3}}\,\mathbf{i} + \frac{s}{\sqrt{3}}\,\mathbf{j} + \frac{s}{\sqrt{3}}\,\mathbf{k}, \quad s \in \Big[-\frac{\sqrt{3}\pi}{4}, \frac{\sqrt{3}\pi}{4}\Big].$$