Calculus II Midterm 1 - Sample

National Central University, Summer 2012, Mar. 31, 2012

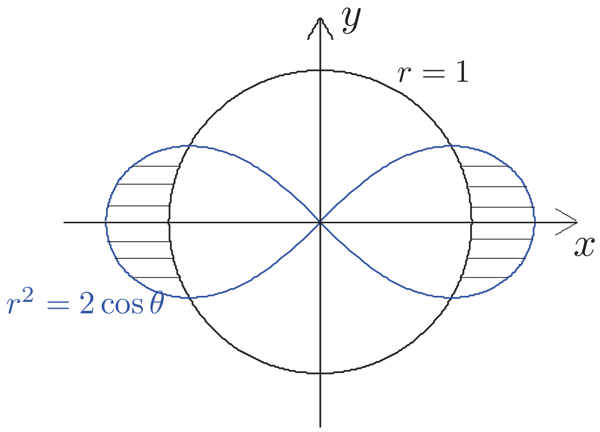
Problem 1. Evaluate the definite integral $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3+2\cos 4x}dx$. $\left(\text{Ans} = \dfrac{\pi}{2\sqrt{5}}\right)$

Problem 2. Find all $\alpha \in \mathbb{R}$ so that the improper integral $\displaystyle\int_1^\infty \frac{1}{x\big[\ln(1+x)\big]^\alpha}dx$ is convergent. $\big(\text{Ans} : \alpha > 1\big)$

Problem 3. Show that $\displaystyle\int_1^e (\ln x)^n dx = e\sum_{k=0}^{n-2}(-1)^k\frac{n!}{(n-k)!} + (-1)^{n-1}n!$.
Hint: Use integration by parts to show that

$$\int_1^e (\ln x)^n dx = e - n\int_1^e (\ln x)^{n-1}dx.$$

Problem 4. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).



1. Find the area of R. $\left(\text{Ans} = \sqrt{3} - \dfrac{\pi}{3}\right)$
2. Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \dfrac{\pi}{6}$. $\big(\text{Ans} = 0\big)$
3. Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

   (a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

   $$y^4 + a(x)y^2 + b(x) = 0 \tag{1}$$

   for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$. $\Big(\text{Ans} : a(x) = 2(x^2+1), b(x) = x^4 - 2x^2\Big)$

(b) Solving (1), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \Big[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\Big].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \big[d(x) - (1-x^2)\big]dx$. $\Big(\text{Ans} = 0\Big)$

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx$, and then find I.

$$\left(\text{Ans}: \begin{array}{l} \displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx = \pi\Big(\frac{3\sqrt{2}}{2} - \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}\Big), \\ \displaystyle I = \pi\Big(\frac{1}{2}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}} + \sqrt{3} - \frac{\sqrt{2}}{3} - \frac{4}{3}\Big). \end{array}\right)$$

4. Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis. $\Big(\text{Ans} = 6\pi\big(2-\sqrt{3}\big)\Big)$

Problem 5. (15%) Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \tan^{-1}\frac{t}{\sqrt{1-t^2}}\mathbf{i} + \sin^{-1}t\,\mathbf{j} + \cos^{-1}t\,\mathbf{k}, \quad t \in \Big[-1, 0.5\Big],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

$$\Big(\text{Ans}: \mathbf{r}_1 = \mathbf{r}_1(s) = \frac{s}{\sqrt{3}}\mathbf{i} + \frac{s}{\sqrt{3}}\mathbf{j} + \big(\frac{\pi}{2} - \frac{s}{\sqrt{3}}\big)\mathbf{k}, \quad s \in \Big[-\frac{\sqrt{3}\pi}{2}, \frac{\sqrt{3}\pi}{6}\Big]\Big)$$