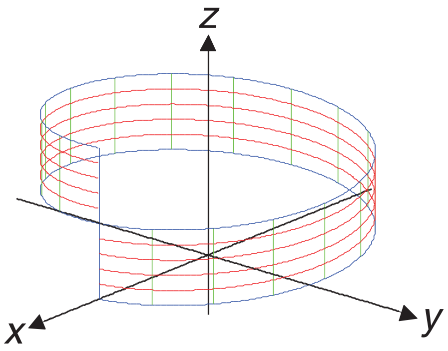
Calculus II Final

National Central University, Spring 2012, Jun. 21, 2012

Problem 1. (15%) Let Σ be part of the cylinder $x^2 + y^2 = 1$ which is bounded by the helix $\vec{r}_1(t) = (\cos t, \sin t, t)$, $0 \le t \le 4\pi$, and a line segment $\vec{r}_2(t) = (1, 0, t)$, $0 \le t \le 4\pi$.



Compute $\displaystyle\iint_\Sigma x^3 dS$.

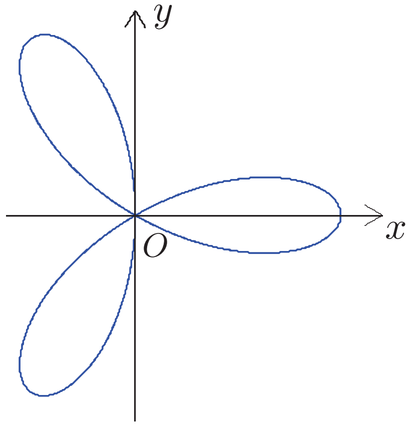
Sol: Σ can be parametrized by

$$\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z), \qquad 0 \le \theta \le 2\pi, \theta \le z \le 2\pi + \theta.$$

Note that $\|\vec{r}_\theta \times \vec{r}_z\| = 1$. Therefore,

$$\begin{aligned}\iint_\Sigma x^3 dS &= \int_0^{2\pi}\int_\theta^{2\pi+\theta} \cos^3\theta dz d\theta = \int_0^{2\pi} (2\pi + \theta - \theta)\cos^3\theta d\theta \\ &= 2\pi \int_0^{2\pi} \cos^3\theta d\theta = 2\pi \int_0^{2\pi} \cos\theta(1 - \sin^2\theta) d\theta \\ &= 2\pi(\sin\theta - \sin^3\theta)\Big|_{\theta=0}^{\theta=2\pi} = 0.\end{aligned}$$

Problem 2. Let C be the polar curve with polar representation $r = \cos 3\theta$, $0 \le \theta \le \pi$.



1. (15%) Use the area formula

$$A = \frac{1}{2}\oint_C x dy - y dx$$

to compute the area enclosed by the Cardioid.

2. (15%) Let $\vec{F}(x,y) = (4x, 3xy)$ be a vector field on the plane. Use Green's theorem to compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{r}(t) = \cos 3t(\cos t, \sin t)$, $0 \le t \le \pi$.

3. (10%) Let C_1 be part of the Cardioid C with $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$. Compute the line integral $\int_C \vec{G} \cdot d\vec{r}$ where $\vec{G}(x,y) = (e^x y, e^x)$.

Sol:

1. A parametrization of the curve C is $\vec{r}(t) = (x(t), y(t)) = (\cos 3t \cos t, \cos 3t \sin t)$, $0 \le t \le \pi$. Since $x(t) = \dfrac{\cos 4t + \cos 2t}{2}$ and $y(t) = \dfrac{\sin 4t - \sin 2t}{2}$, $x'(t) = -2\sin 4t - \sin 2t$ and $y'(t) = 2\cos 4t - \cos 2t$. Therefore,

$$\begin{aligned}\oint_C xdy - ydx &= \int_0^\pi \Big[\frac{\cos 4t + \cos 2t}{2}\big(2\cos 4t - \cos 2t\big) + \frac{\sin 4t - \sin 2t}{2}\big(2\sin 4t + \sin 2t\big)\Big]dt \\ &= \int_0^\pi \Big[\cos^2 4t + \frac{1}{2}\cos 4t \cos 2t - \frac{1}{2}\cos^2 2t + \sin^2 4t - \frac{1}{2}\sin 4t \sin 2t - \frac{1}{2}\sin^2 2t\Big]dt \\ &= \int_0^\pi \Big[\frac{1}{2} + \frac{1}{2}\cos 6t\Big]dt = \frac{\pi}{2};\end{aligned}$$

thus the area enclosed by C is $\pi/4$.

2. $\vec{F}(x,y) = (4x, 3xy)$. By Green's theorem,

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \iint_R 3ydA = \int_0^\pi \int_0^{\cos 3\theta} 3r^2 \sin\theta r dr d\theta \\ &= \int_0^\pi \cos^3 3\theta \sin\theta d\theta = \frac{1}{4}\int_0^\pi (\cos 9\theta + 3\cos 3\theta)\sin\theta d\theta \\ &= \frac{1}{8}\int_0^\pi \big[\sin 10\theta - \sin 8\theta + 3\big(\sin 4\theta - \sin 2\theta\big)\big]d\theta = 0.\end{aligned}$$

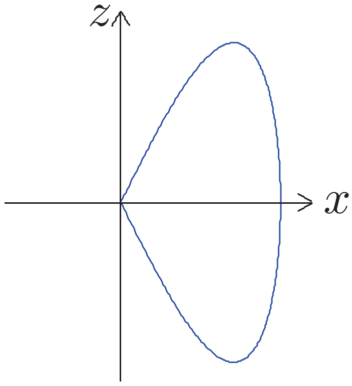
3. Note that $\vec{G} = \nabla g$ if $g(x,y) = ye^x$. Therefore, by the fundamental theorem of the line integral, we find that

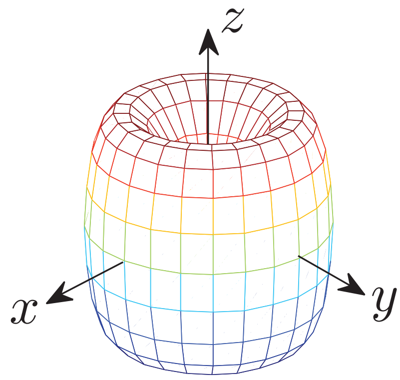
$$\int_{C_1} \vec{G} \cdot d\vec{r} = g(\vec{r}(\frac{2\pi}{3})) - g(\vec{r}(\frac{\pi}{3})) = g(-\frac{1}{2}, \frac{\sqrt{3}}{2}) - g(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \sqrt{3}e^{-\frac{1}{2}}.$$

Problem 3. Let D be the solid given by

$$(x,y,z) = \Phi(u,v,w) = (\cos v \cos u, \cos v \sin u, w \sin 2v), \quad 0 \le u \le 2\pi, -\frac{\pi}{2} \le v \le \frac{\pi}{2}, 0 \le w \le 1,$$

whose surface Σ is obtained by rotating the curve $\vec{r}(t) = (\cos t, \sin 2t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, on the xz-plane about the z-axis.





1. (10%) Compute the volume of D.
2. (10%) Let $\vec{r}(u,v) = \Phi(u,v,1)$. Then $\vec{r}(u,v)$ with $(u,v) \in [0,2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a parametrization of Σ. Compute $\vec{r}_u \times \vec{r}_v$, as well as $\|\vec{r}_u \times \vec{r}_v\|$.
3. (5%) There are two unit normal vectors $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ and $-\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ at each point $\vec{r}(u,v)$ on Σ. Determine which one is compatible with the outward pointing orientation.
4. (20%) Let $\vec{F}(x,y,z) = (x,y,0)$. Compute the flux integral $\iint_\Sigma \vec{F} \cdot \vec{\mathrm{N}}\, dS$ by the definition of surface integral.
5. (5%) Use the divergence theorem to compute the surface integral $\iint_\Sigma \vec{F} \cdot \vec{\mathrm{N}}\, dS$, where $\vec{\mathrm{N}}$ is the outward point unit normal to Σ.

Sol:

1. Since

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & 2w \cos 2v \\ 0 & 0 & \sin 2v \end{vmatrix} = \sin 2v \sin v \cos v,$$

the volume of D is

$$\iiint_D dV = \int_0^1 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du dv dw = \int_0^1 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \frac{\sin^2 2v}{2} du dv dw$$
$$= \pi \int_{-\pi/2}^{\pi/2} \frac{1-\cos 4v}{2} dv = \frac{\pi^2}{2}.$$

2. Since

$$\vec{r}_u(u,v) = (-\cos v \sin u, \cos v \cos u, 0),$$
$$\vec{r}_v(u,v) = (-\sin v \cos u, -\sin v \sin u, 2\cos 2v),$$

we find that

$$(\vec{r}_u \times \vec{r}_v)(u,v) = (2\cos 2v \cos v \cos u, 2\cos 2v \cos v \sin u, \sin v \cos v)$$

and

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4\cos^2 2v \cos^2 v + \sin^2 v \cos^2 v} = \cos v\sqrt{4\cos^2 2v + \sin^2 v}.$$

3. At $(x, y, z) = (1, 0, 0)$, the corresponding $(u, v) = (0, 0)$. Since $\vec{\mathrm{N}}(1, 0, 0) = (1, 0, 0)$, and

$$\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}(0, 0) = (1, 0, 0).$$

Therefore, $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ is compatible with the outward pointing orientation.

4. The flux integral $\iint_\Sigma \vec{F} \cdot \vec{\mathrm{N}}\, dS$ can be computed by

$$\begin{aligned}
&\iint_R \vec{F}\big(\vec{r}(u, v)\big) \cdot (\vec{r}_u \times \vec{r}_v)(u, v)\, dudv \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \big(2\cos 2v \cos^2 v \cos^2 u + 2\cos 2v \cos^2 v \sin^2 u\big) dudv \\
&= 4\pi \int_{-\pi/2}^{\pi/2} \cos 2v \cos^2 v dv = 4\pi \int_{-\pi/2}^{\pi/2} \cos 2v \frac{1 + \cos 2v}{2} dv \\
&= 2\pi \int_{-\pi/2}^{\pi/2} \big(\cos 2v + \frac{1 + \cos 4v}{2}\big) dv = \pi^2.
\end{aligned}$$

5. By the divergence theorem,

$$\iint_\Sigma \vec{F} \cdot \vec{\mathrm{N}}\, dS = \iiint_D \mathrm{div}\, \vec{F}\, dV = 2\iiint_D dV = 2 \times \frac{\pi^2}{2} = \pi^2.$$

Problem 4. Complete the following.

1. (10%) Suppose f is a scalar function which has continuous partial derivatives. Use the divergence theorem to show that

$$\iiint_D \frac{\partial f}{\partial y}(x, y, z) dV = \iint_\Sigma f(x, y, z) \mathrm{N}_2(x, y, z) dS\,, \tag{0.1}$$

where Σ is the boundary of D (or D is enclosed by Σ), and $\vec{\mathrm{N}} = (\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3)$ is the outward pointing unit normal to Σ.

2. (10%) Use (0.1) to compute

$$\iint_\Sigma y^2 e^z dS\,,$$

where Σ is the sphere $x^2 + y^2 + z^2 = 9$. You can use the formula $\int xe^{ax} dx = (\frac{x}{a} - \frac{1}{a^2})e^{ax}$ to reduce the computation.

Solution:

(a) Let $\vec{F}(x,y,z) = (0, f(x,y,z), 0)$. By the divergence theorem,

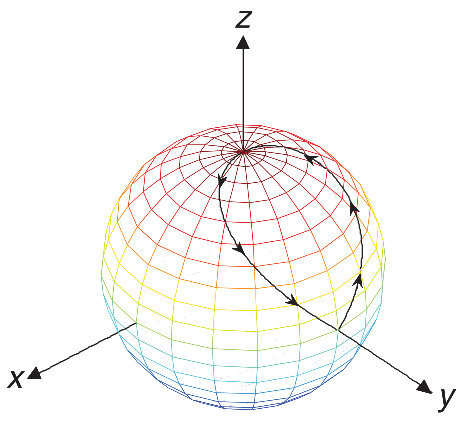
$$\iint_\Sigma f(x,y,z) n_2(x,y,z)\,dS = \iint_\Sigma \vec{F}(x,y,z)\cdot \vec{n}(x,y,z)\,dS = \iiint_D \operatorname{div} \vec{F}(x,y,z)\,dV$$
$$= \iiint_D \frac{\partial f}{\partial y}(x,y,z)\,dV\,.$$

(b) On the sphere $x^2+y^2+z^2=9$, the outward point normal vector $\vec{n}(x,y,z) = \frac{1}{3}(x,y,z)$. Therefore, by (0.1) (with $f(x,y,z)=3ye^z$ in mind),

$$\iint_\Sigma y^2 e^z\,dS = \iint_\Sigma 3ye^z \frac{y}{3}\,dS = \iiint_D \frac{\partial}{\partial y}(3ye^z)\,dV = \int_0^3\int_0^{2\pi}\int_0^\pi 3e^{\rho\cos\phi}\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$
$$= 3\int_0^3\int_0^{2\pi} -\rho e^{\rho\cos\phi}\Big|_{\phi=0}^{\phi=\pi} d\theta\,d\rho$$
$$= 6\pi\int_0^3 (\rho e^\rho - \rho e^{-\rho})\,d\rho$$
$$= 6\pi(\rho-1)e^\rho\Big|_{\rho=0}^{\rho=3} - 6\pi(-\rho-1)e^{-\rho}\Big|_{\rho=0}^{\rho=3}$$
$$= 12\pi(e^3+2e^{-3})\,.$$

Problem 5. Let C be a smooth curve parametrized by

$$\vec{r}(t) = (\cos t\sin t, \sin t\sin t, \cos t)\,, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}\,.$$



1. (10%) Show that the corresponding curve of $\vec{r}(t)$ on $\theta\phi$-plane consists of two line segments L_1 and L_2 given by
$$L_1 = \Big\{(\theta,\phi)\,\Big|\,\theta=\phi\,, 0\le\phi\le\frac{\pi}{2}\Big\},\quad L_2 = \Big\{(\theta,\phi)\,\Big|\,\theta=\pi-\phi\,, 0\le\phi\le\frac{\pi}{2}\Big\}.$$

2. (10%) Plot L_1 and L_2 on the $\theta\phi$-plane. The curve C divides the unit sphere into two parts, and let Σ be the part with smaller area. Identify the corresponding region of Σ on $\theta\phi$-plane.

3. (15%) Find the surface area of Σ.

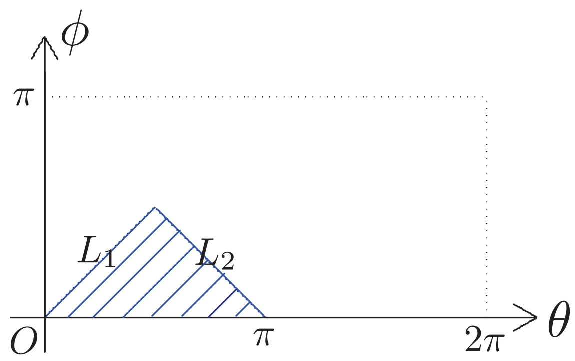
4. (20%) Let $\overrightarrow{F}(x,y,z) = (y,-x,0)$ be a vector field in the space. Compute the line integral $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ by the definition of the line integral.

5. (20%) Use Stokes's Theorem to find the line integral $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$.

Solution:

1. Each point $\overrightarrow{r}(t)$ of C corrsponds to a point $(\theta(t), \phi(t))$ in the $\theta\phi$-plane. Let C_1 be parametrized by $\overrightarrow{r}_1(t) = \overrightarrow{r}(t)$ with $0 \le t \le \frac{\pi}{2}$, and C_2 be parametrized by $\overrightarrow{r}_2(t) = \overrightarrow{r}(t)$ with $-\frac{\pi}{2} \le t \le 0$. For points with $0 \le t \le \frac{\pi}{2}$, $\cos t = \cos\phi$ implies $t = \phi$, and hence $\cos\theta\cos\phi = \cos t \sin t$ and $\sin\theta\sin\phi = \sin t \sin t$ imply $\phi = \theta$. Therefore, C_1 corresponds to the curve $\theta = \phi(= t)$, $0 \le \phi \le \frac{\pi}{2}$. For C_2, since $-\frac{\pi}{2} \le t \le 0$, $\cos\phi = \cos t$ implies $\phi = -t$, and hence $\cos\theta\cos\phi = \cos t \sin t$ and $\sin\theta\sin\phi = \sin t \sin t$ imply $\theta + \phi = \pi$. Therefore, C_2 corresponds to the curve $\theta + \phi = \pi$, $0 \le \phi \le \frac{\pi}{2}$.

2. Let (x,y,z) be the point corresponds to $(\theta,\phi) = (\pi/2, \pi/4)$. This point belongs to Σ, and locates inside the triangle T formed by L_1, L_2 and θ-axis. Therefore, Σ corresponds to T and is plotted as follows.



3. The area of Σ is

$$\int_0^{\frac{\pi}{2}} \int_{\phi}^{\pi-\phi} \sin\phi \, d\theta d\phi = \pi - 2\,.$$

4. Since $\overrightarrow{r}'(t) = (\cos^2 t - \sin^2 t, 2\sin t\cos t, -\sin t)$. By the definition of the line integral,

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t \sin t, -\cos t \sin t, 0) \cdot (\cos^2 t - \sin^2 t, 2\sin t \cos t, -\sin t) dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\sin^2 t \cos^2 t - \sin^4 t - 2\cos^2 t \sin^2 t\right] dt \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t dt = -\frac{\pi}{2}\,.
\end{aligned}$$

5. $\text{curl}\overrightarrow{F}(x,y,z) = (0,0,-2)$. By the Stokes theorem,

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds &= \iint_\Sigma (0,0,-2)\cdot \overrightarrow{\mathrm{N}}\, dS = -2\int_0^{\frac{\pi}{2}} \int_\phi^{\pi-\phi} \sin\phi\cos\phi d\theta d\phi \\
&= -\int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin 2\phi d\phi \\
&= \left[\frac{\pi}{2}\cos 2\phi - \phi\cos 2\phi + \frac{1}{2}\sin 2\phi\right]\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\
&= -\frac{\pi}{2}.
\end{aligned}$$