Calculus II Final Sample

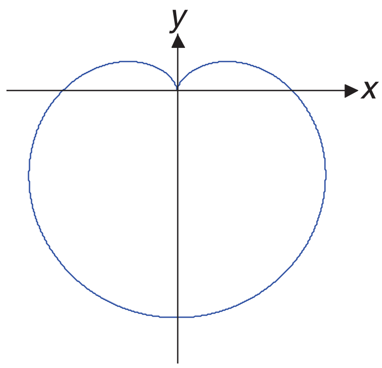
National Central University, Spring 2012, Jun. 17, 2012

Problem 1. Complete the following.

1. Find $\iint_\Sigma x dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

2. Find the area of the part of the cylinder $x^2 + y^2 = 2ay$ that lies outside the cone $z^2 = x^2 + y^2$.

Problem 2. Let C the Cardioid with polar representation $r = (1 - \sin\theta)$, $0 \le \theta \le 2\pi$.



1. Use the area formula

$$A = \frac{1}{2}\oint_C x dy - y dx$$

to compute the area enclosed by the Cardioid by computing the line integral directly.

2. Let $\vec{F}(x, y) = (3xy, 2y)$ be a vector field on the plane. Use Green's theorem to compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$.

3. Let C_1 be part of the Cardioid C with $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. Use the fundamental theorem of line integrals to compute the line integral $\int_C \vec{G} \cdot d\vec{r}$ where $\vec{G}(x, y) = (ye^{xy}, xe^{xy} + 2y)$.

Problem 3. Complete the following.

1. Suppose f is a scalar function which has continuous partial derivatives. Use the divergence theorem to show that

$$\iiint_D \frac{\partial f}{\partial x}(x, y, z) dV = \iint_\Sigma f(x, y, z) \mathrm{N}_1(x, y, z) dS\,, \tag{0.1}$$

where Σ is the boundary of D (or D is enclosed by Σ), and $\vec{\mathrm{N}} = (\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3)$ is the outward pointing unit normal to Σ.

2. Let $\mathbb{S}^2$ denote the unit sphere centered at the origin. Use (0.1) to compute

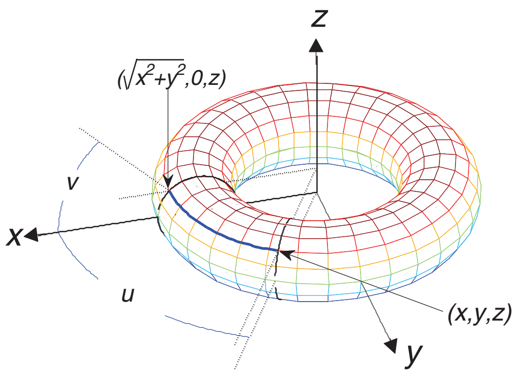
$$\iint_{\mathbb{S}^2} x^2 e^{2z}\, dS\,.$$

You can use the formula $\displaystyle\int xe^{ax}dx = (\frac{x}{a} - \frac{1}{a^2})e^{ax}$ to reduce the computation.

Problem 4. Let D be the solid given by

$$\begin{aligned}(x,y,z) &= \Phi(u,v,w)\\ &= \big((2+w\cos v)\cos u, (2+w\cos v)\sin u, w\sin v\big)\,, \quad 0\le u\le 2\pi\,, 0\le v\le 2\pi\,, 0\le w\le 1\end{aligned}$$

whose surface $\mathbb{T}^2$ is a torus obtained by rotating the curve $\vec{r}(t) = (2+\cos t, \sin t)$, $0\le\theta\le 2\pi$, on the xz-plane about the z-axis.



1. Compute the volume of D.

2. Let $\vec{r}(u,v) = \Phi(u,v,1)$. Then $\vec{r}(u,v)$ with $(u,v)\in[0,2\pi]\times\Big[-\dfrac{\pi}{2},\dfrac{\pi}{2}\Big]$ is a parametrization of Σ. Compute $\vec{r}_u\times\vec{r}_v$, as well as $\|\vec{r}_u\times\vec{r}_v\|$.

3. There are two unit normal vectors $\dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$ and $-\dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$ at each point $\vec{r}(u,v)$ on Σ. Determine which one is compatible with the outward pointing orientation.

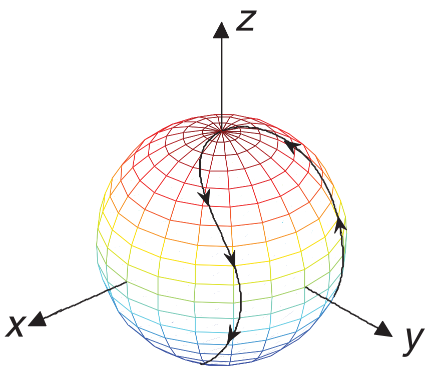
4. Let $\vec{F}(x,y,z) = (\ln(x^2+y^2), \ln(x^2+y^2), \ln(x^2+y^2))$. Use the divergence theorem to compute the surface integral $\displaystyle\iint_{\mathbb{T}^2}\vec{F}\cdot\vec{\mathrm{N}}\,dS$, where $\vec{\mathrm{N}}$ is the outward point unit normal to $\mathbb{T}^2$.

Problem 5. Let C be a smooth curve on the unit sphere parametrized by

$$\vec{r}\ (t) = (\cos(\sin t)\sin t, \sin(\sin t)\sin t, \cos t)\,, \qquad 0\le t\le 2\pi\,.$$

1. Show that the corresponding curve of $\vec{r}\ (t)$ on $\theta\phi$-plane consists of two curves C_1 and C_2 given by

$$C_1 = \big\{(\theta,\phi)\,\big|\,\theta = \sin\phi\,, 0\le\phi\le\pi\big\}, \quad C_2 = \big\{(\theta,\phi)\,\big|\,\theta = \pi - \sin\phi\,, 0\le\phi\le\pi\big\}.$$



2. Plot C_1 and C_2 on the $\theta\phi$-plane. The curve C divides the unit sphere into two parts, and let Σ be the part containing the point $(0, 1, 0)$. Identify the corresponding region of Σ on $\theta\phi$-plane.

3. Find the surface area of Σ.

4. Let $\vec{F}(x, y, z) = (y, -x, 0)$ be a vector field in the space. Compute the line integral $\oint_C \vec{F} \cdot \vec{T}\, ds$ by the definition of the line integral.

5. Use Stokes's Theorem to find the line integral $\oint_C \vec{F} \cdot \vec{T}\, ds$.