Calculus Review 1

National Central University, Spring semester 2012

Problem 1. Compute the following limits.

(1) $\displaystyle\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{x}$ (2) $\displaystyle\lim_{x\to 0^+} \frac{\sqrt[3]{1+x}-1}{1-\cos\sqrt{x}}$.

Problem 2. Complete the following.

(1) Let f and g be two functions, and $f(0)=g(0)=0$ for some number $a\in\mathbb{R}$. Suppose that f and g are differentiable at 0, and $g'(0)\neq 0$. Show that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{f'(0)}{g'(0)}\,.$$

The same conclusion can be drawn if the limit is changed to the right-hand limit or the left-hand limit, as long as f and g are differentiable from the right or the left at 0.

(2) Use (1) to compute the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{\sin x}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2}$.

(3) Suppose that f is twice continuously differentiable. Use (1) to show that

$$\lim_{h\to 0}\frac{f(a+3h)-4f(a)+3f(a-h)}{h^2}=6f''(a)\,.$$

Caution: For (2b) and (3), you cannot simply assign $g(x)=x^2$ or $g(h)=h^2$ since $g'(0)=0$ which is not allowed in order to apply (1).

Problem 3. Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x^{\frac{3}{2}}\sin\dfrac{1}{\sqrt{x}} & \text{if } x\neq 0\,,\\ 0 & \text{if } x=0\,.\end{cases}$ Find the derivatives of f.

Problem 4. Suppose that $f:(0,\infty)\to\mathbb{R}$ and $g:\mathbb{R}\to(0,\infty)$ are two strictly increasing, differentiable functions satisfying

$$f(g(x))=x\quad\forall\,x\in\mathbb{R}\,,\qquad g(f(x))=x\quad\forall\,x\in(0,\infty)\,,$$

and $f(ab)=f(a)+f(b)$ for all $a,b>0$.

(1) Show that $f'(1)g'(0)=1$.

(2) Show that $xf'(x)g'(x)=g(x)$ for all $x>0$.

Problem 5. Suppose that x and y satisfy the relation $1+x=\sin(xy^2)$.

(1) Find $\dfrac{dy}{dx}$ using the implicit differentiation.

(2) Find the tangent line to the curve at the point $(-1/2, \sqrt{7\pi/3})$.

Problem 6. Suppose that $f(x) = -\big(2 + \frac{\pi}{4}\big)\sin x + x\cos x$ is defined on the interval $\big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$.

(1) Show that

$$|f(x) - f(y)| \le \big(1 + \frac{\pi}{2}\big)\frac{\sqrt{2}}{2}|x - y| \qquad \forall\, x, y \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big].$$

(2) Sketch the graph of f defined in (2) with the information of

(a) intercepts;

(b) interval of increase and decrease;

(c) extreme values and critical points; and

(d) concavity and inflection points.

Problem 7. Show that $x^3+x+1 = 0$ has exactly one solution, and use Newton's method to compute the approximated solution x_2 with the initial guess $x_0 = 0$.

Problem 8. Find an anti-derivative of $g(x) = x\sin x$.
Hint: Check the derivative of f defined in Problem 1 (2).