Calculus Quiz 15

1. (5 pts)

a. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)dx$. Compare your result to the actual value. Can you explain the discrepancy?

b. Let f be a polynomial with $\deg f = 3$ or lower, which defined on $[a, b]$. Show that the Simpson's Rule gives the exact value of $\int_a^b f(x)dx$ [*Hint*: it suffice to show the result when there are two subintervals ($n = 2$), since for a larger even number of subintervals the sum of exact estimates is exact.]

Sol.

a. Let $f(x) = \cos(\pi x)$, and $\Delta x = \dfrac{20-0}{10} = 2$. The by Trapezoidal Rule,

$$\begin{aligned} T_{10} &= \frac{2}{2}\big[f(0) + 2\big(f(2) + f(4) + \cdots + f(18)\big) + f(20)\big] \\ &= \cos 0 + 2\big(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi\big) + \cos 20\pi \\ &= 1 + 2 \cdot 18 + 1 = 20 \end{aligned}$$

The actual value of the integral is

$$\begin{aligned} \int_0^{20} \cos(\pi x)dx &= \frac{1}{\pi}\int_0^{20\pi} \cos u du, \text{ by letting } u=\pi x \Rightarrow du=\pi dx \\ &= \frac{1}{\pi}\sin u\Big|_{u=0}^{u=20\pi} = \frac{1}{\pi}\big(\sin 20\pi - \sin 0\big) = 0 \end{aligned}$$

The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value $f(2n) = \cos 2n\pi = 1$.

b. Let $f(x) = Ax^3 + Bx^2 + Cx + D$ and let $\Delta x = h = \dfrac{b-a}{2}$. Then we set $x_0 = a,\ x_1 = a + h,\ x_2 = b$. Without loss of generality, we may shift our graph of f to the left such that $x_1 = 0$, that is, by letting $y = x - a - h$ and $g(y) = f(x - a - h)$, then

$$\int_{-h}^{h} g(y)dy = \int_a^b f(x)dx$$

By Simpson's Rule, we have that

$$\begin{aligned} S_2 &= \frac{h}{3}\big(g(-h) + 4g(0) + g(h)\big) \\ &= \frac{h}{3}\big(- Ah^3 + Bh^2 - Ch + D + 4D + Ah^3 + Bh^2 + Ch + D\big) \\ &= \frac{2}{3}Bh^3 + 2Dh \end{aligned}$$

The exact value of integral is

$$\begin{aligned} \int_a^b f(x)dx = \int_{-h}^{h} g(y)dy &= \int_{-h}^{h} \big(Ay^3 + By^2 + Cy + D\big)dy \\ &= \Big(\frac{A}{4}y^4 + \frac{B}{3}y^3 + \frac{C}{2}y^2 + Dy\Big)\Big|_{-h}^{h} = \frac{2}{3}Bh^3 + 2Dh \end{aligned}$$

which is coincide with S_2, the prove is complete.

$\square$

2. (5 pts) *The extension of factorial to non-integer values.*
By the fact that the improper integral $\int_0^\infty t^{x-1}e^{-t}dt$ is convergent for $x > 0$. We define it as a function of x, called the Gamma function $\Gamma(x)$.

a. Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, in particular $\Gamma(n+1) = n!$ when n is positive integer.

b. Have known that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Find $\Gamma\Big(\frac{3}{2}\Big)$.

Sol.

a. For $x \geq 1$, then $x - 1 \geq 0$ and thus

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt = \lim_{b\to\infty}\int_0^b t^x e^{-t}dt \\ &= \lim_{b\to\infty}\Big[- t^x e^{-t}\Big|_0^b + \int_0^b xt^{x-1}e^{-t}dt\Big], \text{ by letting } \begin{smallmatrix} u=t^x, & dv=e^{-t}dt \\ du=xt^{x-1}dt, & v=-e^{-t} \end{smallmatrix} \\ &= -\lim_{b\to\infty} b^x e^{-b} + x\lim_{b\to\infty}\int_0^b t^{x-1}e^{-t}dt \\ &= x\Gamma(x) \end{aligned}$$

where the limit $\lim_{b\to\infty} b^x e^{-b} = 0$ is due L'Hospital's Rule. In fact,

$$\lim_{b\to\infty} b^x e^{-b} = \lim_{b\to\infty} x b^{x-1} e^{-b} = \cdots = \lim_{b\to\infty} \Big(\prod_{i=0}^{[x]} (x-i)\Big) b^{x-[x]-1} e^{-b}$$
$$= \lim_{b\to\infty} \frac{x(x-1)\cdots(x-[x])}{b^{1+[x]-x} e^b} = 0$$

For $0 < x < 1$, then $x - 1 < 0$ and hence $t^{x-1}e^{-t}$ is singular at $t = 0$. Thus, similar to above argument,

$$\Gamma(x+1) = \lim_{a\to 0}\lim_{b\to\infty} \int_a^b t^x e^{-t} dt = \lim_{a\to 0}\lim_{b\to\infty} \Big[-t^x e^{-t}\Big|_a^b + x\int_a^b t^{x-1} e^{-t} dt \Big]$$
$$= \lim_{a\to 0}\lim_{b\to\infty} \Big(a^x e^{-a} - b^x e^{-b}\Big) + x\int_0^\infty t^{x-1} e^{-t} dt$$
$$= \lim_{a\to 0} a^x e^{-a} - \lim_{b\to\infty} b^x e^{-b} + x\Gamma(x) = x\Gamma(x)$$

In particular, for $n \in \mathbb{N} \setminus \{0\}$,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots$$
$$= n(n-1)\cdots 3\cdot 2\cdot \Gamma(1) = n!\int_0^\infty e^{-t} dt$$
$$= n! \lim_{b\to\infty} \int_0^b e^{-t} dt = n! \lim_{b\to\infty} \big(1 - e^{-b}\big) = n!.$$

b.

$$\Gamma\Big(\frac{3}{2}\Big) = \int_0^\infty t^{\frac{3}{2}-1} e^{-t} dt = \int_0^\infty t^{\frac{1}{2}} e^{-t} dt$$
$$= 2\int_0^\infty s^2 e^{-s^2} ds, \text{ by letting } s=t^{\frac{1}{2}} \Rightarrow ds = \tfrac{1}{2} t^{-\frac{1}{2}} dt = \tfrac{dt}{2s} \Rightarrow 2s\,ds = dt$$
$$= 2\lim_{b\to\infty} \Big[-\frac{1}{2} s e^{-s^2}\Big|_0^b + \frac{1}{2}\int_0^b e^{-s^2} ds \Big], \text{ by letting } \begin{matrix} u=s, & dv = se^{-s^2} ds \\ du = ds, & v = -\frac{1}{2}e^{-s^2} \end{matrix}$$
$$= -\lim_{b\to\infty} b e^{-b^2} + \lim_{b\to\infty} \int_0^b e^{-s^2} ds = -\lim_{b\to\infty} \frac{b}{e^{b^2}} + \int_0^\infty e^{-s^2} ds$$
$$= -\lim_{b\to\infty} \frac{1}{2be^{b^2}} + \frac{\pi}{2} = \frac{\pi}{2}$$

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