

Calculus Quiz 10

1. (5 pts)

a. Let $F(x) = \int_1^x \int_{\sqrt{t}}^{t^2} \frac{\sqrt{1+u^4}}{u} du dt$. Find $F''(2)$.

b. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}, \text{ for all } x > 0$$

Sol.

a. By FTC, we know that

$$F'(x) = \int_{\sqrt{x}}^{x^2} \frac{\sqrt{1+u^4}}{u} du,$$

and

$$\begin{aligned} F''(x) &= \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot 2x - \frac{\sqrt{1+(\sqrt{x})^4}}{\sqrt{x}} \cdot \left(-\frac{1}{2\sqrt{x}}\right) \\ &= \frac{2\sqrt{1+x^8}}{x} + \frac{\sqrt{1+x^2}}{2x} \end{aligned}$$

$$\text{Hence } F''(2) = \sqrt{257} + \frac{\sqrt{5}}{4}$$

b. Differentiate the equality with respect to x on both side, then by FTC, we get

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$$

Substituting $x = a$ into the equality, we have that

$$6 = 2\sqrt{a} \Rightarrow a = 9$$

□

2. (5 pts)

a. If f is continuous on $[0, \pi]$, show that

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

[Hint: Taking $u = \pi - x$]

b. Use a. to evaluate the integral

$$\int_0^\pi x \sin^2 x \cos^4 x dx$$

Sol.

a.

$$\begin{aligned}\int_0^\pi x f(\sin x) dx &= \int_\pi^0 (\pi - u) f(\sin(\pi - u)) (-du), \text{ by letting } u=1-x \Rightarrow du=-dx \\ &= \int_0^\pi (\pi - u) f(\sin u) du = \pi \int_0^\pi f(\sin u) du - \int_0^\pi u f(\sin u) du \\ &= \pi \int_0^\pi f(\sin u) du - \int_0^\pi x f(\sin x) dx\end{aligned}$$

Thus we get $2 \int_0^\pi x f(\sin x) dx = \pi \int_0^\pi f(\sin x) dx$ and therefore

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

b. By ra., we have that

$$\begin{aligned}\int_0^\pi x \sin^2 x \cos^4 x dx &= \int_0^\pi x \sin^2 x (1 - \sin^2 x)^2 dx = \int_0^\pi \sin^2 x (1 - \sin^2 x)^2 dx \\ &= \int_0^\pi \sin^2 x \cos^4 x dx = \int_0^\pi (\sin x \cos x)^2 \cos^2 x dx \\ &= \frac{1}{4} \int_0^\pi \sin^2 2x \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{8} \int_0^\pi \sin^2 2x (1 + \cos 2x) dx \\ &= \frac{1}{8} \int_0^\pi \sin^2 2x dx + \frac{1}{8} \int_0^\pi \sin^2 2x \cos 2x dx \\ &= \frac{1}{8} \int_0^\pi \frac{1 - \cos 4x}{2} dx + \frac{1}{16} \int_0^\pi u^2 du, \text{ by letting } u = \sin 2x \Rightarrow du = 2 \cos 2x dx \\ &= \frac{1}{16} \int_0^\pi dx - \frac{1}{2} \int_0^\pi \cos 4x dx \\ &= \frac{\pi}{16} - \frac{1}{8} \int_0^{4\pi} \cos v dv, \text{ by letting } v = 4x \Rightarrow dv = 4 dx \\ &= \frac{\pi}{16} - \frac{1}{8} \sin v \Big|_{v=0}^{v=4\pi} = \frac{\pi}{16}\end{aligned}$$

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