Calculus Quiz 8

1. (5 pts) Sketch the curve defined by the function $f(x) = \dfrac{x^3}{x^2-1}$.

Sol. Observe that the denominator of f vanishes at $x = \pm 1$, so the vertical asymptote for $f(x)$ are $x = \pm 1$. Also, the division algorithm shows that

$$f(x) = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

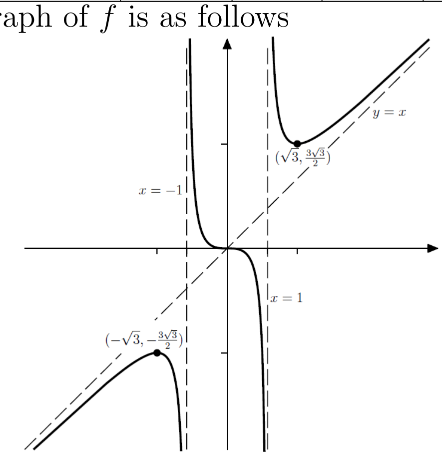
So the line $y = x$ is a slant asymptote for f. Compute the first and second derivative of f, we get

$$f'(x) = \frac{x^4-3x^2}{(x^2-1)^2}, \qquad f''(x) = \frac{2(x^3+3x)}{(x^2-1)^3}$$

Set $f'(x) = 0$, we have that $x^4 - 3x^2 = x^2(x^2-3) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{3}$. Note that $f''(0) = 0$, $f''(\pm\sqrt{3}) = \pm\frac{3\sqrt{3}}{2}$, so f has local maximum at $x = -\sqrt{3}$ and local minimum at $x = \sqrt{3}$. Furthermore, set $f''(x) = 0$, we get $x^3 + 3x = x(x^2+3) = 0 \Rightarrow x = 0$. Together with the observation that f is an odd function, that is, $f(-x) = -f(x)$. We can conclude that f has an inflection point at $x = 0$. Thus we have following table

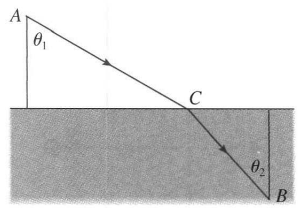
x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
f'	+	0	−	undef.	−	0	−	undef.	−	0	+
f''	−	−	−	undef.	+	0	−	undef.	+	+	+
f	↷	max.	⤵	undef.	↳	inf.pt	⤵	undef.	↳	inf.pt.	⤴

And the graph of f is as follows



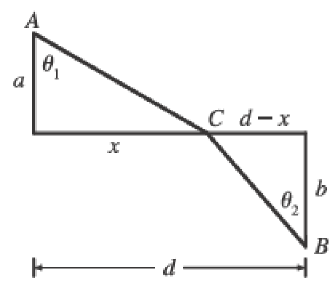
□

2. (5 pts) Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to *Fermat's Principle*, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimize the time taken.



Show that $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$ where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as *Snell's Law*.

Proof. Consider the following figure, The total time is



$$\begin{aligned} T(x) &= (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) \\ &= \frac{\sqrt{a^2 = x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}, \ 0 < x < d \end{aligned}$$

Then

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}.$$

Note that

$$T''(x) = \frac{a^2}{v_1(a^2+x^2)^{\frac{3}{2}}} + \frac{b^2}{v_2\big(b^2+(d-x)^2\big)^{\frac{3}{2}}} > 0, \ \forall\, x \in (0,d)$$

Hence the minimum occurs when $T'(x) = 0$, that is,

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

□