Calculus Quiz 6

1. (5 pts) Suppose that the edge lengths x, y and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}$$

Find the rates at which the box's **a.** volume V, **b.** surface area S, and **c.** diagonal length $\ell = \sqrt{x^2+y^2+z^2}$ are changing at the instant when $x = 4,\ y = 3$, and $z = 2$.

Sol.

a. Since $V = xyz$, then

$$\frac{dV}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}$$

Hence $\left.\dfrac{dV}{dt}\right|_{(x,y,z)=(4,3,2)} = 2 \text{ m}^3/\text{sec}$

b. Since $S = 2xy + 2yz + 2xz$, then

$$\frac{dS}{dt} = 2(y+z)\frac{dx}{dt} + 2(x+z)\frac{dy}{dt} + 2(x+y)\frac{dz}{dt}$$

Hence $\left.\dfrac{dS}{dt}\right|_{(x,y,z)=(2,3,4)} = 0 \text{ m}^2/\text{sec}$

c. Since $\ell = \sqrt{x^2+y^2+z^2}$, then

$$\begin{aligned}\frac{d\ell}{dt} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt} \\ &= \frac{x}{\ell}\frac{dx}{dt} + \frac{y}{\ell}\frac{dy}{dt} + \frac{z}{\ell}\frac{dz}{dt}\end{aligned}$$

Since $\ell\big|_{(x,y,z)=(4,3,2)} = \sqrt{29}$, so $\left.\dfrac{d\ell}{dt}\right|_{(x,y,z)=(4,3,2)} = 0$ m/sec.

□

2. (5 pts)

a. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b,\ 0 \le x \le 1$.

b. Prove that the function $f(x) = x^{101}+x^{51}+x+1$ has neither a local maximum nor a local minimum.

Proof.

a. Since $f(x) = x^a(1-x)^b$, so

$$\begin{aligned}f'(x) &= ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} \\ &= x^{a-1}(1-x)^{b-1}(a - ax - bx)\end{aligned}$$

Thus, $f'(x) = 0 \Leftrightarrow x = 0,\ 1$ or $x = \dfrac{a}{a+b}$. Note that $f(1) = f(0) = 0$, and

$$f\left(\frac{a}{a+b}\right) = \frac{a^a}{(a+b)^a}\frac{b^b}{(a+b)^b}. = \frac{a^a b^b}{(a+b)^{a+b}} > 0$$

since $a, b > 0$. Hence the absolute maximum of $f(x)$ is $\dfrac{a^a b^b}{(a+b)^{a+b}}$.

b. Suppose there is a local maximum of local minimum of f, then there exists $\alpha \in \mathbb{R}$ such that $f'(\alpha) = 0$. Thus,

$$f'(\alpha) = 101\alpha^{100} + 51\alpha^{50} + 1 = 0 \Rightarrow 101\alpha^{100} + 51\alpha^{50} = -1$$

which leads to a contradiction. Therefore f has neither a local maximum nor a local minimum.

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