Calculus Quiz 5

1. (5 pts)

a. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2,\ f(2) = 3,\ f'(1) = 4,\ f'(2) = 5$, and $f'(3) = 6$. Find $F'(1)$.

b. Find the points on the curve $xy^2 + yx^2 = 2$ where the tangent line is horizontal or vertical.

Sol.

a. Since $F(x) = f(xf(xf(x)))$, then

$$\begin{aligned} F'(x) &= f'(xf(xf(x))) \cdot \frac{d}{dx}\Big(xf(xf(x))\Big) \\ &= f'(xf(xf(x))) \cdot \Big[f(xf(x)) + xf'(xf(x)) \cdot \frac{d}{dx}\Big(xf(x)\Big)\Big] \\ &= f'(xf(xf(x))) \cdot \Big[f(xf(x)) + xf'(xf(x)) \cdot \big(f(x) + xf'(x)\big)\Big] \end{aligned}$$

Thus

$$\begin{aligned} F'(1) &= f'(f(f(1))) \cdot \Big[f(f(1)) + f'(f(1)) \cdot \big(f(1) + f'(1)\big)\Big] \\ &= f'(f(2)) \cdot \Big[f(2) + f'(2) \cdot (2+4)\Big] \\ &= f'(3) \cdot \Big[3 + 5 \cdot 6\Big] = 6 \cdot 33 = 198 \end{aligned}$$

b. By differentiating implicitly the equation with respect to x on both side, we get

$$y^2 + 2xy\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy = 0$$

Thus

$$\frac{dy}{dx} = -\frac{y^2+2xy}{x^2+2xy}, \qquad \frac{dx}{dy} = -\frac{x^2+2xy}{y^2+2xy}$$

Note that the horizontal (resp. vertical) tangent occur when $\frac{dy}{dx} = 0$ (resp. $\frac{dx}{dy} = 0$). For $\frac{dy}{dx} = 0$, then $y^2 + 2xy = y(y+2x) = 0 \Rightarrow y = 0$ or $y = -2x$. It easy to see that y can not be zero. By substituting $y = -2x$ into the equation, we get $4x^3 - 2x^3 = 2x^3 = 2 \Rightarrow x^3 = 1 \Rightarrow x = 1$. Hence the horizontal tangent for the curve occur at the point $(1, -2)$. Similarly, for $\frac{dx}{dy} = 0$, then $x^2+2xy = x(x+2y) = 0 \Rightarrow x = 0$ or $x = -2y$. By the same reason, we know that $x = -2y$

and thus we get $y = 1$. Hence the vertical tangent for the curve occur at the point $(-2, 1)$.

$\square$

2. (5 pts)

a. Show that the nth derivative of $\cos^3 x$ is

$$\frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right]$$

b. Show that the implicit function defined by quadratic form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + k = 0$ has first and second derivative as

$$\frac{dy}{dx} = -\frac{ax + by + d}{bx + cy + e}, \qquad \frac{d^2y}{dx^2} = \frac{\Delta}{(bx + cy + e)^3}$$

where

$$\Delta = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & k \end{vmatrix}$$

Proof.

a. Let $f(x) = \cos x$. Observe that

$$f'(x) = -\sin x,\ f''(x) = -\cos x,\ f'''(x) = \sin x$$

Hence $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$. Since $\cos 3x = 4\cos^3 x - 3\cos x$, so

$$\cos^3 x = \frac{1}{4}\big(\cos 3x - 3\cos x\big) = \frac{1}{4}\big(f(3x) - 3f(x)\big)$$

Hence

$$\begin{aligned} \frac{d^n}{dx^n}\cos^3 x &= \frac{1}{4}\big(f^{(n)}(3x) - 3f^{(n)}(x)\big) \\ &= \frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right] \end{aligned}$$

b. Differentiating the equation with respect to x on both side, we get

$$\begin{aligned} & 2ax + 2by + 2bx\frac{dy}{dx} + 2cy\frac{dy}{dx} + 2d + 2e\frac{dy}{dx} = 0 \\ \Rightarrow\ & (2ax + 2by + 2d) + (2bx + 2cy + ed)\frac{dy}{dx} = 0 \end{aligned}$$

This shows that

$$\frac{dy}{dx} = -\frac{ax + by + d}{bx + cy + e}$$

Also, since $k = -ax^2 - 2bxy - cy^2 - 2dx - 2ey$, then

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = -\frac{d}{dx}\Big(\frac{ax+by+d}{bx+cy+e}\Big) \\
&= -\frac{\big(a+b\frac{dy}{dx}\big)(bx+cy+e)-(ax+by+d)\big(b+c\frac{dy}{dx}\big)}{(bx+cy+e)^2} \\
&= -\frac{\big(a-b\frac{ax+by+d}{bx+cy+e}\big)(bx+cy+e)-(ax+by+d)\big(b-c\frac{ax+by+d}{bx+cy+e}\big)}{(bx+cy+e)^2} \\
&= \frac{1}{(bx+cy+e)^3}\Big[-cd^2+2bde-ae^2+2b^2dx-2acdx+ab^2x^2-a^2cx^2 \\
&\qquad +2b^2ey-2acey+2b^3xy-2abcxy+b^2cy^2-ac^2y^2\Big] \\
&= \frac{1}{(bx+cy+e)^3}\Big[-cd^2+2bde-ae^2-ac(ax^2+2bxy+cy^2+2dx+2ey) \\
&\qquad +b^2(ax^2+2bxy+cy^2+2dx+2ey)\Big] \\
&= \frac{ack-b^2k+2bde-cd^2-ae^2}{(bx+cy+e)^3} \\
&= \frac{\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & k \end{vmatrix}}{(bx+cy+e)^3} = \frac{\Delta}{(bx+cy+e)^3}
\end{aligned}$$

$\square$