Calculus Quiz 1

1. (5 pts)

a. Evaluate the limit $\lim_{x\to\frac{1}{n}^+} x\left[\frac{1}{x}\right]$ for $n\in\mathbb{N}$, and $\lim_{x\to 0^+} x\left[\frac{1}{x}\right]$.

b. Is there a number a such that

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value of a and the value of the limit.

Sol.

a.

$$\lim_{x\to\frac{1}{n}^+} x\left[\frac{1}{x}\right] = \Big(\lim_{x\to\frac{1}{n}^+} x\Big)\Big(\lim_{x\to\frac{1}{n}^+}\left[\frac{1}{x}\right]\Big) = \frac{1}{n}\Big(\lim_{y\to n^+}[y]\Big) = \frac{n-1}{n}$$

On the other hand, since $\frac{1}{x}-1\le\left[\frac{1}{x}\right]\le\frac{1}{x}$, so

$$1-x\le x\left[\frac{1}{x}\right]\le 1$$

Since $\lim_{x\to 0^+}(1-x)=\lim_{x\to 0^+}1=1$. By Squeeze Theorem, we have that $\lim_{x\to 0^+} x\left[\frac{1}{x}\right]=1$.

b. Note that

$$\frac{3x^2+ax+a+3}{x^2+x-2}=\frac{3x^2+ax+a+3}{(x+2)(x-1)}$$

Hence the limit $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exists if and only if $x+2$ divides $3x^2+ax+a+3$. Let $f(x)=3x^2+ax+a+3$, then the limit exists if and only if $f(-2)=0$, that is, $12-2a+a+3=0\Rightarrow a=15$. In this case,

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}=\lim_{x\to -2}\frac{3(x+2)(x+3)}{(x+2)(x-1)}=\lim_{x\to -2}\frac{3(x+3)}{(x-1)}=-1$$

□

2. (5 pts)

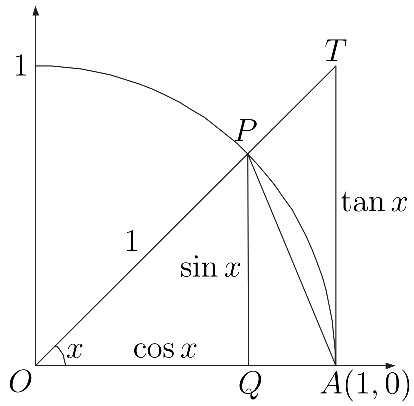
a. Show that $|\sin x|\le|x|\le|\tan x|$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

b. Using **a.** to prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

c. Derive a formula for area of regular n-gon inscribed in circle with radius r and show that the area of the circle is πr^2.

Sol.

a. For $0 < x < \frac{\pi}{2}$. Consider the graph as follows



It is clear that

$$\text{area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT$$

This immediately implies that

$$0 < \sin x < x < \tan x$$

For $-\frac{\pi}{2} < x < 0$, let $y = -x$, then $0 < y < \frac{\pi}{2}$, and thus we have $\sin y < y < \tan y$. That is,

$$0 < -\sin x = \sin(-x) < -x < \tan(-x) = -\tan x$$

and hence $0 > \sin x > x > \tan x$. Note that $\sin 0 = \tan 0 = 0$. Therefore, we get

$$|\sin x| \leq |x| \leq |\tan x|, \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

b. For $0 \leq x < \frac{\pi}{2}$, we have that $\sin x \leq x \leq \tan x$. Dividing $\sin x$ on both side, we get

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

By taking reciprocal, we have that

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

Since $\lim_{x \to 0^+} \cos x = \lim_{x \to 0^+} 1 = 1$. By Squeeze Theorem, we have that $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$. For $-\frac{\pi}{2} < x \leq 0$, then $\sin x \geq$

$x \geq \tan x$. By argument similar to that for positive x, we have that $\lim_{x \to 0^-} \dfrac{\sin x}{x} = 1$. Hence

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

c. By connecting each vertices of n-gon with center of circle, we get n identical isosceles triangles with length r and included angle $\dfrac{2\pi}{n}$. Thus the area $A(n)$ of regular n-gon inscribed in circle is

$$A(n) = \frac{nr^2}{2} \sin \frac{2\pi}{n}$$

We can approaching the area of circle by taking limit of $A(n)$ as $n \to \infty$. Therefore, the area A of the circle with radius r is

$$\begin{aligned} A = \lim_{n \to \infty} A(n) = r^2 \lim_{n \to \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \pi r^2 \lim_{n \to \infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \\ = \pi r^2 \lim_{x \to 0} \frac{\sin x}{x}, \quad \text{by letting } x = \tfrac{2\pi}{n} \Rightarrow x \to 0 \text{ as } n \to \infty \\ = \pi r^2 \end{aligned}$$

□