Calculus I Final, Sample

National Central University, Summer 2011, Aug. 25, 2011

Problem 1. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles. (This is Exercise Problem 66 in Section 6.2).

Problem 2. Let $p > 0$. Show that $\frac{t}{2} + \ln \mathrm{C}_p \geq (p-1)\ln t$ for $t > 2p$, where $\mathrm{C}_p = \left[2p\right]^{p-1} e^{-p}$.

Problem 3. Complete the following.

(1) Show that the (improper) integral $\int_0^1 t^{a-1}(1-t)^{b-1}dt$ is convergent for all $a, b > 0$.

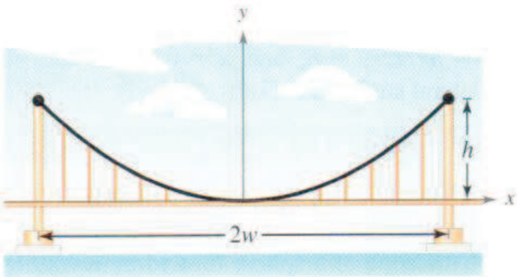
(2) Let $\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$. By (2) $\beta(a,b)$ is defined for all $a, b > 0$. Show that $a\beta(a,b) = (b-1)\beta(a+1,b-1)$ for all $a > 0$ and $b > 1$. In particular, also show that

$$\frac{1}{m+n+1} \cdot \frac{1}{\beta(m+1,n+1)} = \binom{m+n}{n} = \frac{(m+n)!}{m! \times n!} \qquad \forall\, m,n \in \mathbb{N}.$$

Problem 4. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from its lowest point to its highest point and let $2w$ represent the total span of the bridge (see figure). Show that the length L of the cable is given by

$$L = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx\,,$$

and evaluate L.



Problem 5. Evaluate the definite integral $\int_0^{2\pi} \frac{1}{3+2\cos x}dx$. $\left(\text{The answer is } \frac{2\pi}{\sqrt{5}}\right)$.

Problem 6. The goal of this problem is to find the indefinite integral $\int \frac{1}{(1+x^3)^{\frac{1}{3}}}dx$. Complete the following.

(1) By the substitution of variable $x^3 = \tan^2\theta$, show that

$$\int \frac{1}{(1+x^3)^{\frac{1}{3}}}dx = \frac{2}{3}\int \frac{1}{\cos\theta \sin^{\frac{1}{3}}\theta}d\theta\,.$$

(2) Then make another substitution of variable $u^3 = \sin\theta$, show that

$$\int \frac{1}{\cos\theta \sin^{\frac{1}{3}}\theta} d\theta = \int \frac{3u}{(1-u^6)} du .$$

(3) Using the technique of integrating rational functions by partial fractions, find the indefinite integral in (1) and then express the result in terms of x so that one obtains

$$\begin{aligned}\int \frac{1}{(1+x^3)^{\frac{1}{3}}} dx = {} & \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{2(1+x^{-3})^{\frac{1}{6}}+1}{\sqrt{3}} - \tan^{-1}\frac{2(1+x^{-3})^{\frac{1}{6}}-1}{\sqrt{3}}\right] \\ & + \frac{1}{6}\ln\left[(1+x^{-3})^{-\frac{2}{3}} + (1+x^{-3})^{-\frac{1}{3}} + 1\right] \\ & - \frac{1}{3}\ln\left[1-(1+x^{-3})^{-\frac{1}{3}}\right] + C .\end{aligned}$$