Basic Mathematics（基礎數學）MA1015A Midterm Exam I

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Problem 1. (10%) An integer x has property P provided that

"for all integers a, b, whenever x divides ab, x divide a or x divides b".

Explain what it means to say that x does not have property P.

Solution. By assumption, x has property P if (and only if)

$$(\forall\,(a,b)\in\mathbb{Z}\times\mathbb{Z})\big(x|(ab)\Rightarrow(x|a)\vee(x|b)\big)\,.$$

Therefore,

$$\begin{aligned}
x \text{ does not have property } P &\Leftrightarrow\ \sim(x \text{ has property } P)\\
&\Leftrightarrow\ \sim\Big[(\forall\,(a,b)\in\mathbb{Z}\times\mathbb{Z})\big(x|(ab)\Rightarrow(x|a)\vee(x|b)\big)\Big]\\
&\Leftrightarrow (\exists(a,b)\in\mathbb{Z}\times\mathbb{Z})\Big[x|(ab)\wedge\sim\big((x|a)\vee(x|b)\big)\Big]\\
&\Leftrightarrow (\exists(a,b)\in\mathbb{Z}\times\mathbb{Z})\Big[x|(ab)\wedge x\nmid a\wedge x\nmid b\big)\Big]\,;
\end{aligned}$$

thus x does not have property P means that there are two integers a, b such that x divides ab but x does not divide both a and b. □

Problem 2. (20%) We define a prime to be average provided it is the average of two different prime numbers $\big($for example, $7=\dfrac{11+3}{2}$ is average$\big)$. Consider the following propositions:

P: Every prime greater than 3 is average.

Q: Every even number other than 2 can be written as $x+y$, where x, y are primes, and possibly $x=y$ (for example, $4=2+2$, $6=3+3$, $8=5+3$).

R: Every even number greater than 6 can be written as the sum of two different prime numbers.

Prove that $R\Leftrightarrow P\wedge Q$.

Proof. First we write P,Q,R as logic statements:

$$\begin{aligned}
\mathrm{P}&\equiv(\forall\,p>3, p\text{ prime})(\exists\,q,r\text{ primes})(q\neq r\wedge 2p=q+r)\,,\\
\mathrm{Q}&\equiv(\forall\,n\in\mathbb{N}\backslash\{1\})(\exists\,q,r\text{ primes})(2n=q+r)\,,\\
\mathrm{R}&\equiv(\forall\,n\in\mathbb{N}\backslash\{1,2,3\})(\exists\,q,r\text{ primes})(q\neq r\wedge 2n=q+r)\,.
\end{aligned}$$

"$\Rightarrow$" Assume that R holds.

(R $\Rightarrow$ Q): It suffices to show that if $n = 2, 3$, there exist prime numbers q, r such that $2n = q+r$. Nevertheless, $2 \cdot 2 = 2 + 2$ and $2 \cdot 3 = 3 + 3$, so R $\Rightarrow$ Q.

(R $\Rightarrow$ R): Let $p > 3$ be a prime number. In particular, $p \in \mathbb{N}\backslash\{1, 2, 3\}$. Then R implies that there exists prime numbers q and r such that $q \neq r$ and $2p = q + r$. Therefore, R $\Rightarrow$ P.

"$\Leftarrow$" Assume that P and Q hold. Let $n \in \mathbb{N}\backslash\{1, 2, 3\}$ be given. By Q, there exist prime numbers q and r such that $2n = q + r$.

(a) if $q \neq r$, then $q \neq r$ and $2n = q + r$.

(b) if $q = r$, then $n = q$ is a prime number. since $n > 3$, by P there exist prime numbers q_1 and r_1 such that $q_1 \neq r_1$ and $2n = 2q = q_1 + r_1$.

In either cases, there exist prime numbers q and r such that $q \neq r$ and $2n = q + r$. Therefore, R holds. □

Problem 3. (15%) Show (by contradiction) that there do not exist prime numbers a, b, c such that $a^3 + b^3 = c^3$.

Proof. Suppose that there exist prime numbers a, b, c such that $a^3 + b^3 = c^3$. We note that c cannot be 2 since if a, b are also prime numbers, then $a^3 + b^3 > 2^3$. Since 2 is only one even prime number and $c \neq 2$, we find that $a^3 + b^3$ must be an odd number. Therefore, one and only one of a, b is 2. W.L.O.G. we assume that $b = 2$. Then $a^3 + 8 = c^3$ or equivalently, $a^3 = (c - 2)(c^2 + 2c + 4)$.

Note that $c^2 + 2c + 4 > c - 2$ since

$$c^2 + 2c + 4 - (c - 2) = c^2 + c + 6 = \left(c + \frac{1}{2}\right)^2 + \frac{23}{4} > 0\,.$$

Since a is a prime number, there are two factorizations of a^3: $1 \cdot a^3$ or $a \cdot a^2$. Therefore, either $c-2 = 1$ or $c - 2 = a$ since $a^3 > 1$ and $a^2 > a$.

1. $c - 2 = 1$: Then $c = 3$ and $a^3 = c^2 + 2c + 4 = 19$ which is impossible.

2. $c - 2 = a$: Then $c = a + 2$ and

$$a^3 = c^2 + 2c + 4 = (a + 2)^2 + 2(a + 2) + 4 = a^2 + 6a + 12\,;$$

thus $a^3 - a^2 - 6a - 12 = 0$ which implies that a is a factor of 12. Therefore, $a = 2$ (which is excluded by assumption) or $a = 3$ which is impossible since $3^3 - 3^2 - 6 \cdot 3 - 12 \neq 0$.

Therefore, there are no prime numbers a, b, c satisfying $a^3 + b^3 = c^3$. □

Problem 4. (10%) For non-zero integers a and b, an integer n is called a common multiple of a and b if a divides n and b divides n. We say that the positive integer m is the least common multiple of a and b, written as $\text{lcm}(a, b)$, if

(i) m is a common multiple of a and b, and

(ii) if n is a positive common multiple of a and b, then $m \leqslant n$.

Show that $\operatorname{lcm}(a,b) \cdot \gcd(a,b) = ab$ if a, b are natural numbers.

Proof. Let $a, b \in \mathbb{N}$ and $d = \gcd(a,b)$. Then $a = dm$ and $b = dn$ for some $m, n \in \mathbb{N}$ and $\gcd(m,n) = 1$. Note that $\dfrac{ab}{\gcd(a,b)} = dmn$; thus $\dfrac{ab}{\gcd(a,b)}$ is a common multiple of a and b which implies that

$$\frac{ab}{\gcd(a,b)} \geqslant \operatorname{lcm}(a,b)\,. \tag{$\star$}$$

Next we prove that any common multiple of a and b is not less than $\dfrac{ab}{\gcd(a,b)}$. Suppose that c is a positive common multiple of a and b. Then $c = ka = kdm$ for some $k \in \mathbb{N}$. Since b also divides c, we find that $(dn) \mid (kdm)$ which implies that $n \mid km$. By the fact that $\gcd(m,n) = 1$, we must have $n \mid k$. Therefore, $k = n\ell$ for some $\ell \in \mathbb{N}$. In other words, if c is a positve common multiple of a and b, then $c = kdm = \ell(dmn)$. Therefore, any common multiple of a and b is not less than $\dfrac{ab}{\gcd(a,b)}$. In particular,

$$\operatorname{lcm}(a,b) \geqslant \frac{ab}{\gcd(a,b)}\,. \tag{$\star\star$}$$

Combining $(\star)$ and $(\star\star)$, we conclude that $\operatorname{lcm}(a,b) = \dfrac{ab}{\gcd(a,b)}$. □

Problem 5. (10%) Which of the following statements is true?

1. $(\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})(x < y^2)$.
2. $(\exists\, y \in \mathbb{R})(\forall\, x \in \mathbb{R})(x < y^2)$.

Explain your answer.

Solution. 1. Let $x \in \mathbb{R}$ be given. Choose $y = |x| + 1$. Then by the fact that $2|x| \geqslant x$, we find that

$$y^2 = (|x|+1)^2 = x^2 + 2|x| + 1 > x\,.$$

Therefore, it holds that $(\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})(x < y^2)$.

2. Let $y \in \mathbb{R}$ be given. Then $x = y^2 \geqslant y^2$ which implies that $(\forall\, y \in \mathbb{R})(\exists\, x \in \mathbb{R})(x \geqslant y^2)$ is true. Therefore, $\sim (\forall\, y \in \mathbb{R})(\exists\, x \in \mathbb{R})(x \geqslant y^2)$ is false or equivalently, $(\exists\, y \in \mathbb{R})(\forall\, x \in \mathbb{R})(x < y^2)$ is false. □

Problem 6. (15%) Let A and B be sets. Define an operation of sets Δ by $A\Delta B = (A-B)\cup(B-A)$. Show that $A\Delta B = (A\cup B) - (A\cap B)$. You need to prove the statement logically, as well as using Venn's diagram.

Proof. Let x be an element in the universe. Then by the associative property of set operations,

$$\begin{aligned}
x \in (A\cup B) - (A\cap B) &\Leftrightarrow x \in (A\cup B)\cap(A\cap B)^{\complement} \\
&\Leftrightarrow x \in (A\cup B)\cap(A^{\complement}\cup B^{\complement}) \\
&\Leftrightarrow x \in \big[A\cap(A^{\complement}\cup B^{\complement})\big] \cup \big[B\cap(A^{\complement}\cup B^{\complement})\big] \\
&\Leftrightarrow x \in \big[(A\cap A^{\complement})\cup(A\cap B^{\complement})\big] \cup \big[(B\cap A^{\complement})\cup(B\cap B^{\complement})\big] \\
&\Leftrightarrow x \in \big[\varnothing\cup(A\cap B^{\complement})\big] \cup \big[(B\cap A^{\complement})\cup\varnothing\big] \\
&\Leftrightarrow x \in (A\cap B^{\complement})\cup(B\cap A^{\complement}) \\
&\Leftrightarrow x \in (A-B)\cup(B-A) \Leftrightarrow x \in A\Delta B\,.
\end{aligned}$$

□

Problem 7. (10%) Let X be the universe, and $\mathscr{F}$ be the empty family of subsets of X. Show that

$$\bigcup_{A\in\mathscr{F}} A = \varnothing\,.$$

Proof. Let $x \in X$. Then $x \in \bigcup\limits_{A\in\mathscr{F}} A$ if and only if $(\exists\, A \in \mathscr{F})(x \in A)$. Since $\mathscr{F}$ is the empty family of subsets of X, there is no element in $\mathscr{F}$; thus $(\exists\, A \in \mathscr{F})(x \in A)$ is false. Therefore, $x \in \bigcup\limits_{A\in\mathscr{F}} A$ is false. Since this is false for all given $x \in X$, any element $x \in X$ is not an element of the set $\bigcup\limits_{A\in\mathscr{F}} A$. This implies that $\bigcup\limits_{A\in\mathscr{F}} A = \varnothing$. □

Problem 8. (10%) Suppose that $\mathscr{F} = \{A_k \,|\, i \in \mathbb{N}\}$ is an indexed family of sets such that for all $i, j \in \mathbb{N}$, if $i \leqslant j$, then $A_j \subseteq A_i$. Prove that for all $\ell \in \mathbb{N}$, $\bigcup\limits_{k=\ell}^{\infty} A_k = A_\ell$.

Proof. "$\subseteq$": Let $x \in \bigcup\limits_{k=\ell}^{\infty} A_k$. Then there exists $k \geqslant \ell$ such that $x \in A_k$. Since $A_j \subseteq A_i$ if $i \leqslant j$, we find that $A_k \subseteq A_\ell$; thus $x \in A_\ell$. Therefore, $\bigcup\limits_{k=\ell}^{\infty} A_k \subseteq A_\ell$.

"$\supseteq$": Let $x \in A_\ell$. Then $x \in \bigcup\limits_{k=\ell}^{\infty} A_k$; thus $A_\ell \subseteq \bigcup\limits_{k=\ell}^{\infty} A_k$.

Therefore, $A_\ell = \bigcup\limits_{k=\ell}^{\infty} A_k$. □