§5.3 Countable Sets

Theorem

The union of denumerable denumerable sets is denumerable. In other words, if $\mathcal{F}$ is a denumerable collection of denumerable sets, then $\bigcup_{A\in\mathcal{F}} A$ is denumerable.

Proof.

Let $\mathcal{F} = \{A_i \,|\, i \in \mathbb{N}, A_i \text{ is denumerable}\}$ be an indexed family of denumerable sets, and define $A = \bigcup_{i=1}^{\infty} A_i$. Since A_i is denumerable, we write $A_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}$. Then $A = \{x_{ij} \,|\, i, j \in \mathbb{N}\}$. Let $f : \mathbb{N} \times \mathbb{N} \to A$ be defined by $f(i,j) = x_{ij}$. Then $f : \mathbb{N} \times \mathbb{N} \to A$ is a surjection. Moreover, since $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, there exists a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$; thus $h = f \circ g : \mathbb{N} \to A$ is a surjection which implies that A is countable. Since $A_1 \subseteq A$, A is infinite; thus A is denumerable. □

§5.3 Countable Sets

Corollary

The union of countable countable sets is countable（可數個可數集的聯集是可數的）.

Proof.

By adding empty sets into the family or adding $\mathbb{N}$ into a finite set if necessary, we find that the union of countable countable sets is a subset of the union of denumerable denumerable sets. Since a (non-empty) subset of a countable set is countable, we find that the union of countable countable sets is countable. □

§5.3 Countable Sets

Corollary

The set of rational numbers $\mathbb{Q}$ is countable.

Proof.

Let $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote the collection of positive and negative rational numbers, respectively. We have shown that the set $\mathbb{Q}^+$ is countable. Since $\mathbb{Q}^+ \approx \mathbb{Q}^-$ (between them there exists a one-to-one correspondence $f(x) = -x$), $\mathbb{Q}^-$ is also countable. Therefore, the previous theorem $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable. $\square$

§5.3 Countable Sets

Corollary

1. *If* $\mathfrak{F}$ *is a finite pairwise disjoint family of denumerable sets, then* $\bigcup_{A\in\mathfrak{F}} A$ *is countable.*
2. *If* A *and* B *are countable sets, then* $A \cup B$ *is countable.*
3. *If* $\mathfrak{F}$ *is a finite collection of countable sets, then* $\bigcup_{A\in\mathfrak{F}} A$ *is countable.*
4. *If* $\mathfrak{F}$ *is a denumerable family of countable sets, then* $\bigcup_{A\in\mathfrak{F}} A$ *is countable.*