§5.2 Infinite Sets

Theorem

1. *Even open interval (a, b) is uncountable and has cardinality* $\mathbf{c}$.
2. *The set $\mathbb{R}$ of all real numbers is uncountable and has cardinality* $\mathbf{c}$.

Proof.

1. The function $f(x) = a + (b - a)x$ maps from $(0, 1)$ to (a, b) and is a one-to-one correspondence.
2. Using ①, $(0, 1) \approx \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, the function $f(x) = \tan x$ maps from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$ and is a one-to-one correspondence; thus $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \approx \mathbb{R}$. Since $\approx$ is an equivalence relation, $(0, 1) \approx \mathbb{R}$. □

§5.2 Infinite Sets

Example

The circle with the north pole removed is equivalent to the real line.

Example

The set $A = (0,2) \cup [5,6)$ has cardinality $\mathbf{c}$ since the function $f\colon (0,1) \to A$ given by

$$f(x) = \begin{cases} 4x & \text{if } 0 < x < \frac{1}{2}, \\ 2x + 4 & \text{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

is a one-to-one correspondence from $(0,1)$ to A.

§5.3 Countable Sets

Theorem

Let S be a non-empty set. The following statements are equivalent:

1. *S is countable;*
2. *there exists a surjection $f : \mathbb{N} \to S$;*
3. *there exists an injection $f : S \to \mathbb{N}$.*

Proof.

"① ⇒ ②" First suppose that $S = \{x_1, \cdots, x_n\}$ is finite. Define $f : \mathbb{N} \to S$ by

$$f(k) = \begin{cases} x_k & \text{if } k < n\,, \\ x_n & \text{if } k \geqslant n\,. \end{cases}$$

Then $f : \mathbb{N} \to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f : \mathbb{N} \xrightarrow[\text{onto}]{1-1} S$. □

§5.3 Countable Sets

1. S is countable;
2. there exists a surjection $f\colon \mathbb{N} \to S$;

Proof. (Cont'd).

"①⇐②" W.L.O.G. we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S - \{f(k_1)\} \neq \emptyset$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered principle (**WOP**) of $\mathbb{N}$, N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S - \{f(k_1), f(k_2)\} \neq \emptyset$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(S - \{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$. □

§5.3 Countable Sets

Proof. (Cont'd).

Claim: $f\colon \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is easy to see since $f(k_j) \notin \{f(k_1), f(k_2), \cdots, f(k_{j-1})\}$ for all $j \geqslant 2$. For surjectivity, assume the contrary that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f\colon \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. Therefore, $k \in N_\ell$ and $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ. □

Let $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ be defined by $g(j) = k_j$. Then g is one-to-one and onto; thus $h = g \circ f\colon \mathbb{N} \xrightarrow[\text{onto}]{1-1} S$. □