§5.1 Equivalent Sets; Finite Sets

Lemma

Suppose that A, B, C and D are sets with $A \approx C$ and $B \approx D$.

1. *If A and B are disjoint and C and D are disjoint, then $A \cup B \approx C \cup D$.*
2. $A \times B \approx C \times D$.

Proof.

Suppose that $\phi : A \xrightarrow[onto]{1-1} C$ and $\psi : B \xrightarrow[onto]{1-1} D$.

1. Then $\phi \cup \psi : A \cup B \to C \cup D$ is an one-to-one correspondence.
2. Let $f : A \times B \to C \times D$ be given by
$$f(a,b) = \big(\phi(a), \psi(b)\big) .$$
Then f is an one-to-one correspondence from $A \times B$ to $C \times D$.

□

§5.1 Equivalent Sets; Finite Sets

Definition

For each natural number k, let $\mathbb{N}_k = \{1, 2, \cdots, k\}$. A set S is ***finite*** if $S = \varnothing$ or $S \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. A set S is ***infinite*** if S is not a finite set.

Theorem

For $k, j \in \mathbb{N}$, $\mathbb{N}_j \approx \mathbb{N}_k$ if and only if $k = j$.

Proof.

It suffices to prove the $\Rightarrow$ direction. Suppose that $\phi : \mathbb{N}_k \to \mathbb{N}_j$ is a one-to-one correspondence. W.L.O.G. we can assume that $k \leqslant j$. If $k < j$, then $\phi(\mathbb{N}_k) = \big\{\phi(1), \phi(2), \cdots, \phi(k)\big\} \neq \mathbb{N}_j$ since the number of elements in $\phi(\mathbb{N}_k)$ and $\mathbb{N}_j$ are different. In other words, if $k < j$, $\phi : \mathbb{N}_k \to \mathbb{N}_j$ cannot be surjective. This implies that $\mathbb{N}_k \approx \mathbb{N}_j$ if and only if $k = j$. □

§5.1 Equivalent Sets; Finite Sets

Definition

Let S be a finite set. If $S = \varnothing$, then S has ***cardinal number*** 0 (or ***cardinality*** 0), and we write $\#S = 0$. If $S \approx \mathbb{N}_k$ for some natural number k, then S has ***cardinal number*** k (or ***cardinality*** k), and we write $\#S = k$.

Remark: The cardinality of a set S can also be denoted by $n(S)$, $\overline{\overline{S}}$, $\mathrm{card}(S)$ as well.

Theorem

If A is finite and $B \approx A$, then B is finite.

Lemma

If S is a finite set with cardinality k and x is any object not in S, then $S \cup \{x\}$ is finite and has cardinality $k + 1$.

§5.1 Equivalent Sets; Finite Sets

Lemma

For every $k \in \mathbb{N}$, every subset of $\mathbb{N}_k$ is finite.

Proof.

Let $S = \left\{k \in \mathbb{N} \,\middle|\, \text{the statement "every subset of } \mathbb{N}_k \text{ is finite" holds}\right\}$.

1. There are only two subsets of $\mathbb{N}_1$, namely $\varnothing$ and $\mathbb{N}_1$. Since $\varnothing$ and $\mathbb{N}_1$ are both finite, we have $1 \in S$.
2. Suppose that $k \in S$. Then every subset of $\mathbb{N}_k$ is finite. Since $\mathbb{N}_{k+1} = \mathbb{N}_k \cup \{k+1\}$, every subset of $\mathbb{N}_{k+1}$ is either a subset of $\mathbb{N}_k$, or the union of a subset of $\mathbb{N}_k$ and $\{k+1\}$. By the fact that $k \in S$, we conclude from the previous lemma that every subset of $\mathbb{N}_{k+1}$ is finite.

Therefore, **PMI** implies that $S = \mathbb{N}$. □

§5.1 Equivalent Sets; Finite Sets

Theorem

Every subset of a finite set is finite.

Proof.

Let $A \subseteq B$ and B is a finite set.

1. If $A = \varnothing$, then A is a finite set (and $\#A = 0$).
2. If $A \neq \varnothing$, then $B \neq \varnothing$. Since B is finite, there exists $k \in \mathbb{N}$ such that $B \approx N_k$; thus there exists a one-to-one correspondence $\phi : \mathbb{N}_k \to B$. Therefore, $\phi^{-1}(A)$ is a non-empty subset of $\mathbb{N}_k$, and the previous lemma implies that $\phi^{-1}(A)$ is finite. Since $A \approx \phi^{-1}(A)$, we conclude that A is a finite set. □

§5.1 Equivalent Sets; Finite Sets

Theorem

1. *If A and B are disjoint finite sets, then $A \cup B$ is finite, and*
$$\#(A \cup B) = \#A + \#B\,.$$
2. *If A and B are finite sets, then $A \cup B$ is finite, and*
$$\#(A \cup B) = \#A + \#B - \#(A \cap B)\,.$$
3. *If $A_1, A_2, \cdots, A_n$ are finite sets, then $\bigcup_{k=1}^{n} A_k$ is finite.*

Proof.

1. W.L.O.G., we assume that $A \approx \mathbb{N}_k$ and $B \approx \mathbb{N}_\ell$ for some $k, \ell \in \mathbb{N}$. Let $H = \{k+1, k+2, \cdots, k+\ell\}$. Then $\mathbb{N}_\ell \approx H$ since $\phi(x) = k + x$ is a one-to-one correspondence from $\mathbb{N}_\ell \to \{k+1, k+2, \cdots, k+\ell\}$. Therefore, $A \cup B \approx \mathbb{N}_k \cup H = \mathbb{N}_{k+\ell}$; thus $\#(A \cup B) = \#A + \#B$. □

§5.1 Equivalent Sets; Finite Sets

Proof of $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.

2. Note that $A \cup B$ is the disjoint union of A and $B - A$, where $B - A$ is a subset of a finite set B which makes $B - A$ a finite set. Therefore, $A \cup B$ is finite.

   To see $\#(A \cup B) = \#A + \#B - \#(A \cap B)$, using ① it suffices to show that $\#(B - A) = \#B - \#(A \cap B)$. Nevertheless, note that $B = (B - A) \cup (A \cap B)$ in which the union is in fact a disjoint union; thus ① implies that

$$\#B = \#(B - A) + \#(A \cap B)$$

   or equivalently,

$$\#(B - A) = \#B - \#(A \cap B) .$$

□

§5.1 Equivalent Sets; Finite Sets

Proof.

3. Let $A_1, A_2, \cdots$ be finite sets, and

$$S = \left\{ n \in \mathbb{N} \,\middle|\, \bigcup_{k=1}^{n} A_k \text{ is finite} \right\}.$$

Then $1 \in S$ by assumption. Suppose that $n \in S$. Then $n+1 \in S$ because of ②. **PMI** then implies that $S = \mathbb{N}$. □

§5.1 Equivalent Sets; Finite Sets

Lemma

Let $k \geqslant 2$ be a natural number. For $x \in \mathbb{N}_k$, $\mathbb{N}_k \backslash \{x\} \approx \mathbb{N}_{k-1}$.

Theorem (Pigeonhole Principle - 鴿籠原理)

Let $n, r \in \mathbb{N}$ and $f : \mathbb{N}_n \to \mathbb{N}_r$ be a function. If $n > r$, then f is not injective.

Corollary

If $\#A = n$, $\#B = r$ and $r < n$, then there is no one-to-one function from A to B.

Corollary

If A is finite, then A is not equivalent to any of its proper subsets.

§5.2 Infinite Sets

Recall that a set A is infinite if A is not finite. By the last corollary in the previous section, if a set is equivalent to one of its proper subset, then that set cannot be finite. Therefore, $\mathbb{N}$ is not finite since there is a one-to-one correspondence from $\mathbb{N}$ to the set of even numbers.

The set of natural numbers $\mathbb{N}$ is a set with infinite cardinality. The standard symbol for the cardinality of $\mathbb{N}$ is $\aleph$. There are two kinds of infinite sets, denumerable（無窮可數）sets and uncountable（不可數）sets.

Definition

A set S is said to be ***denumerable*** if $S \approx \mathbb{N}$. For a denumerable set S, we say S has cardinal number $\aleph_0$ (or cardinality $\aleph_0$) and write $\#S = \aleph_0$.

§5.2 Infinite Sets

Example

The set of even numbers and the set of odd numbers are denumerable.

Example

The set $\{p, q, r\} \cup \{n \in \mathbb{N} \mid n \neq 5\}$ is denumerable.

Theorem

The set $\mathbb{Z}$ is denumerable.

Proof.

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even}, \\ \dfrac{1-x}{2} & \text{if } x \text{ is odd}. \end{cases}$$

□

§5.2 Infinite Sets

Theorem

1. *The set $\mathbb{N} \times \mathbb{N}$ is denumerable.*
2. *If A and B are denumerable sets, then $A \times B$ is denumerable.*

Proof.

1. Consider the function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $F(m, n) = 2^{m-1}(2n-1)$. Then $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective.
2. If A and B are denumerable sets, then $A \approx \mathbb{N}$ and $B \approx \mathbb{N}$. Then $A \times B \approx \mathbb{N} \times \mathbb{N}$; thus $A \times B \approx \mathbb{N}$ since $\approx$ is an equivalence relation. $\square$

Definition

A set S is said to be ***countable*** if S is finite or denumerable. We say S is ***uncountable*** if S is not countable.

§5.2 Infinite Sets

Theorem

The open interval $(0,1)$ is uncountable.

Proof.

Assume the contrary that there exists a bijection $f : \mathbb{N} \to (0,1)$. Write $f(k)$ in decimal expansion (十進位展開); that is,

$$
\begin{aligned}
f(1) &= 0.d_{11}d_{21}d_{31}\cdots \\
f(2) &= 0.d_{12}d_{22}d_{32}\cdots \\
\vdots \quad & \quad\ \vdots \\
f(k) &= 0.d_{1k}d_{2k}d_{3k}\cdots \\
\vdots \quad & \quad\ \vdots
\end{aligned}
$$

Here we note that repeated 9's are chosen by preference over terminating decimals; that is, for example, we write $\frac{1}{4} = 0.249999\cdots$ instead of $\frac{1}{4} = 0.250000\cdots$. □

§5.2 Infinite Sets

Proof. (Cont'd).

Let $x \in (0,1)$ be such that $x = 0.d_1 d_2 \cdots$, where

$$d_k = \begin{cases} 5 & \text{if} \quad d_{kk} \neq 5\,, \\ 3 & \text{if} \quad d_{kk} = 5\,. \end{cases}$$

(建構一個 x 使其小數點下第 k 位數與 $f(k)$ 的小數點下第 k 位數不相等). Then $x \neq f(k)$ for all $k \in \mathbb{N}$, a contradiction; thus $(0,1)$ is uncountable. □

Definition

A set S has cardinal number $\mathbf{c}$ (or cardinality $\mathbf{c}$) if S is equivalent to $(0,1)$. We write $\#S = \mathbf{c}$, which stands for ***continuum***.