§4.5 Set Images

Definition

Let $f: A \to B$ be a function, and $X \subseteq A$, $Y \subseteq B$. The ***image*** of X (under f) or ***image set*** of X, denoted by $f(X)$, is the set

$$f(X) = \{y \in B \mid y = f(x) \text{ for some } x \in X\} = \{f(x) \mid x \in X\},$$

and the ***pre-image*** of Y (under f) or the ***inverse image*** of Y, denoted by $f^{-1}(Y)$, is the set

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

Remark: Here are some facts about images of sets that follow from the definitions:

(a) If $a \in D$, then $f(a) \in f(D)$.

(b) If $a \in f^{-1}(E)$, then $f(a) \in E$.

(c) If $f(a) \in E$, then $a \in f^{-1}(E)$.

(d) If $f(a) \in f(D)$ and f is one-to-one, then $a \in D$.

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Theorem

Let $f: A \to B$ be a function. Suppose that C, D are subsets of A, and E, F are subsets of B. Then

1. $f(C \cap D) \subseteq f(C) \cap f(D)$. *In particular, if* $C \subseteq D$, *then* $f(C) \subseteq f(D)$.
2. $f(C \cup D) = f(C) \cup f(D)$.
3. $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$. *In particular, if* $E \subseteq F$, *then* $f^{-1}(E) \subseteq f^{-1}(F)$.
4. $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.
5. $C \subseteq f^{-1}(f(C))$.
6. $f(f^{-1}(E)) \subseteq E$.

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Proof of $f(C \cap D) \subseteq f(C) \cap f(D)$.

Let $y \in f(C \cap D)$. Then there exists $x \in C \cap D$ such that $y = f(x)$. Therefore, $y \in f(C)$ and $y \in f(D)$; thus $y \in f(C) \cap f(D)$. □

Remark: It is possible that $f(C \cap D) \subsetneq f(C) \cap f(D)$. For example, $f(x) = x^2$, $C = (-\infty, 0)$ and $D = (0, \infty)$. Then $C \cap D = \varnothing$ which implies that $f(C \cap D) = \varnothing$; however, $f(C) = f(D) = (0, \infty)$.

Proof of $f(C \cup D) = f(C) \cup f(D)$.

Let $y \in B$ be given. Then

$$\begin{aligned} y \in f(C \cup D) &\Leftrightarrow (\exists\, x \in C \cup D)\big(y = f(x)\big) \\ &\Leftrightarrow (\exists\, x \in C)\big(y = f(x)\big) \vee (\exists\, x \in D)\big(y = f(x)\big) \\ &\Leftrightarrow \big(y \in f(C)\big) \vee \big(y \in f(D)\big) \\ &\Leftrightarrow y \in f(C) \cup f(D)\,. \end{aligned}$$

□

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Proof of $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

Let $x \in A$ be given. Then

$$\begin{aligned} x \in f^{-1}(E \cap F) &\Leftrightarrow f(x) \in E \cap F \\ &\Leftrightarrow \big(f(x) \in E\big) \wedge \big(f(x) \in F\big) \\ &\Leftrightarrow \big(x \in f^{-1}(E)\big) \wedge \big(x \in f^{-1}(F)\big) \\ &\Leftrightarrow x \in f^{-1}(E) \cap f^{-1}(F) . \end{aligned}$$

□

Proof of $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.

Let $x \in A$ be given. Then

$$\begin{aligned} x \in f^{-1}(E \cup F) &\Leftrightarrow f(x) \in E \cup F \\ &\Leftrightarrow \big(f(x) \in E\big) \vee \big(f(x) \in F\big) \\ &\Leftrightarrow \big(x \in f^{-1}(E)\big) \vee \big(x \in f^{-1}(F)\big) \\ &\Leftrightarrow x \in f^{-1}(E) \cup f^{-1}(F) . \end{aligned}$$

□

§4.5 Set Images

Proof of $C \subseteq f^{-1}(f(C))$.

Let $x \in C$. Then $f(x) \in f(C)$; thus $x \in f^{-1}(f(C))$. Therefore, $C \subseteq f^{-1}(f(C))$. □

Remark: It is possible that $C \subsetneq f^{-1}(f(C))$. For example, if $f(x) = x^2$ and $C = [0, 1]$, then $f^{-1}(f(C)) = f^{-1}([0, 1]) = [-1, 1] \supsetneq [0, 1]$.

Proof of $f(f^{-1}(E)) \subseteq E$.

Suppose that $y \in f(f^{-1}(E))$. Then there exists $x \in f^{-1}(E)$ such that $f(x) = y$. Since $x \in f^{-1}(E)$, there exists $z \in E$ such that $f(x) = z$. Then $y = z$ which implies that $y \in E$. Therefore, $f(f^{-1}(E)) \subseteq E$. □

Remark: It is possible that $f(f^{-1}(E)) \subsetneq E$. For example, if $f(x) = x^2$ and $E = [-1, 1]$, then $f(f^{-1}(E)) = f([0, 1]) = [0, 1] \subsetneq [-1, 1]$.

Chapter 5. Cardinality

§5.1 Equivalent Sets; Finite Sets

Definition

Two sets A and B are ***equivalent*** if there exists a one-to-one function from A onto B. The sets are also said to be ***in one-to-one correspondence***, and we write $A \approx B$. In notation,

$$A \approx B \Leftrightarrow (\exists\, f : A \to B)(f \text{ is a bijection}) .$$

If A and B are not equivalent, we write $A \not\approx B$.

Example

The set of even integers is equivalent to the set of odd integers: the function $f(x) = x + 1$ does the job.

Example

The set of even numbers is equivalent to the set of integers: the function $f(x) = \dfrac{x}{2}$ does the job.

§5.1 Equivalent Sets; Finite Sets

Example

The set of natural numbers is equivalent to the set of integers.

Example

For $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$, the open intervals (a, b) and (c, d) are equivalent. Therefore, any two open intervals are equivalent, even when the intervals have different length.

Example

Let $\mathcal{F}$ be the set of all binary sequences; that is, the set of all functions from $\mathbb{N} \to \{0, 1\}$. Then $\mathcal{F} \approx \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. To see this, we define $\phi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ by $\phi(x) \equiv \left\{k \in \mathbb{N} \,\middle|\, x_k = 1\right\}$ for all $x \in \mathcal{F}$. Then ϕ is well-defined and $\phi : \mathcal{F} \xrightarrow[\text{onto}]{1-1} \mathcal{P}(\mathbb{N})$.

§5.1 Equivalent Sets; Finite Sets

Theorem

Equivalence of sets is an equivalence relation on the class of all sets.

Proof.

1. **Reflexivity**: for all sets A, the identity map I_A is an one-to-one correspondence on A.
2. **Symmetry**: Suppose that $A \approx B$; that is, there exists a one-to-one correspondence ϕ from A to B. Then ϕ^{-1} is an one-to-one correspondence from B to A; thus $B \approx A$.
3. **Transitivity**: Suppose that $A \approx B$ and $B \approx C$. Then there exist one-to-one correspondences $\phi : A \xrightarrow[onto]{1-1} B$ and $\psi : B \xrightarrow[onto]{1-1} C$. Then $\psi \circ \phi : A \to C$ is an one-to-one correspondence; thus $A \approx C$.

□