§4.3 Functions that are Onto; One-to-One Functions

Theorem

If $f : A \to B$, $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Theorem

Let $f : A \to B$ and $g : B \to C$ be functions.

1. *If $g \circ f$ is onto C, then g is onto C.*
2. *If $g \circ f$ is one-to-one, then f is one-to-one.*

Proof.

1. Let $c \in C$. Since $g \circ f$ is onto C, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a)$. Then $g(b) = g(f(a)) = (g \circ f)(a) = c$.
2. Suppose that $f(x) = f(y)$. Then $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$, and the injectivity of $g \circ f$ implies that $x = y$.

□

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Remark:

1. In part ① of the theorem above, we cannot conclude that f is also onto B since there might be a proper subset $\widetilde{B} \subsetneq B$ such that $f : A \to \widetilde{B}$, $g : \widetilde{B} \to C$ and $g \circ f$ is onto C. For example, Let $A = B = \mathbb{R}$, $C = \mathbb{R}^+ \cup \{0\}$, and $f(x) = g(x) = x^2$. Then clearly f is not onto B but $g \circ f$ is onto C.

2. In part ② of the theorem above, we cannot conclue that g is one-to-one since it might happen that g is one-to-one on $\mathrm{Rng}(f) \subsetneq B$ but g is not one-to-one on B. For example, let $A = C = \mathbb{R}^+ \cup \{0\}$, $B = \mathbb{R}$, and $f(x) = x^2$, $g(x) = \log(1 + |x|)$. Then clearly g is not one-to-one, but $g \circ f$ is one-to-one.

§4.3 Functions that are Onto; One-to-One Functions

Theorem

If $f : A \to B$ is one-to-one, then every restriction of f is one-to-one.

In the following we consider the function $f \cup g$. Recall that if $\mathrm{Dom}(f) \cap \mathrm{Dom}(g) = \varnothing$, then $(f \cup g)(x) \overset{(\star)}{=} \begin{cases} f(x) & \text{if } x \in \mathrm{Dom}(f), \\ g(x) & \text{if } x \in \mathrm{Dom}(g). \end{cases}$

Theorem

Let $f : A \to C$ and $g : B \to D$ be functions. Suppose that A and B are disjoint sets.

1. *If f is onto C and g is onto D, then $f \cup g : A \cup B \to C \cup D$ is onto $C \cup D$.*
2. *If f is one-to-one, g is one-to-one, and C and D are disjoint, then $f \cup g : A \cup B \to C \cup D$ is one-to-one.*

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Proof.

We note that $f \cup g : A \cup B \to C \cup D$ is a function.

1. Let $y \in C \cup D$. Then $y \in C$ or $y \in D$. W.L.O.G., we can assume that $y \in C$. Since $f : A \to C$ is onto C, there exists $x \in A$ such that $(x, y) \in f$. Using $(\star)$, $(f \cup g)(x) = f(x) = y$. Therefore, $f \cup g$ is onto $C \cup D$.
2. Suppose that $(x_1, y), (x_2, y) \in f \cup g \subseteq (A \times C) \cup (B \times D)$. Then $(x_1, y) \in f$ or $(x_1, y) \in g$. W.L.O.G., we can assume that $(x_1, y) \in f$. Since $f \subseteq A \times C$ and $g \subseteq B \times D$, by the fact that $C \cap D = \varnothing$ we must have $(x_2, y) \in f$ for otherwise $y \in C \cap D$, a contradiction. Now, since $(x_1, y), (x_2, y) \in f$, the injectivity of f then implies that $x_1 = x_2$. □

§4.4 Inverse Functions

Recall that the inverse of a relation $f : A \to B$ is the relation f^{-1} satisfying

$$yf^{-1}x \quad \Leftrightarrow \quad xfy \quad \Leftrightarrow \quad (x, y) \in f \quad \Leftrightarrow \quad y = f(x)\,.$$

This relation is a function, called the inverse function of f, if the relation itself is a function with certain domain.

Definition

A function $f : A \to B$ is said to be a ***one-to-one correspondence*** if f is a bijection.

§4.4 Inverse Functions

Theorem

Let $f: A \to B$ be a function.

1. *f^{-1} is a function from $\mathrm{Rng}(f)$ to A if and only if f is one-to-one.*
2. *If f^{-1} is a function, then f^{-1} is one-to-one.*

Proof.

1. "$\Rightarrow$" If $(x_1, y), (x_2, y) \in f$, then $(y, x_1), (y, x_2) \in f^{-1}$. Since f^{-1} is a function, we must have $x_1 = x_2$. Therefore, f is one-to-one.
   "$\Leftarrow$" If $(y, x_1), (y, x_2) \in f^{-1}$, then $(x_1, y), (x_2, y) \in f$, and the injectivity of f implies that $x_1 = x_2$. Therefore, by the fact that $\mathrm{Rng}(f) = \mathrm{Dom}(f^{-1})$, f^{-1} is a function with domain $\mathrm{Rng}(f)$.
2. Suppose that f^{-1} is a function, and $(y_1, x), (y_2, x) \in f^{-1}$. Then $(x, y_1), (x, y_2) \in f$ which, by the fact that f is a function, implies that $y_1 = y_2$. Therefore, f^{-1} is one-to-one. □

§4.4 Inverse Functions

Corollary

The inverse of a one-to-one correspondence is a one-to-one correspondence.

Theorem

Let $f: A \to B$, $g: B \to A$ be functions. Then

1. $g = f^{-1}$ *if and only if* $g \circ f = I_A$ *and* $f \circ g = I_B$ *(if and only if* $f = g^{-1}$*).*
2. *If* f *is surjective, and* $g \circ f = I_A$*, then* $g = f^{-1}$*.*
3. *If* f *is injective, and* $f \circ g = I_B$*, then* $g = f^{-1}$*.*

Recall that "If $C = \text{Rng}(f)$ and $f^{-1} : C \to A$ is a function, then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_C$". Therefore, the $\Rightarrow$ direction in ① has already been proved.

§4.4 Inverse Functions

Proof.

We first prove the following two claims:

(a) If $g \circ f = I_A$, then $f^{-1} \subseteq g$. (b) If $f \circ g = I_B$, then $g \subseteq f^{-1}$.

To see (a), let $(y, x) \in f^{-1}$ be given. Then $(x, y) \in f$ or $y = f(x)$. Since $(g \circ f) = I_A$, we must have

$$g(y) = g(f(x)) = (g \circ f)(x) = I_A(x) = x$$

or equivalently, $(y, x) \in g$. Therefore, $f^{-1} \subseteq g$.

To see (b), let $(y, x) \in g$ be given. Then $x = g(y)$; thus the fact that $(f \circ g) = I_B$ implies that

$$f(x) = f(g(y)) = (f \circ g)(y) = I_B(y) = y$$

or equivalently, $(x, y) \in f$. Therefore, $(y, x) \in f^{-1}$; thus $g \subseteq f^{-1}$.

1. "$\Rightarrow$" Done.

   "$\Leftarrow$" This direction is a direct consequence of the claims. □

§4.4 Inverse Functions

Proof. (Cont'd).

2. Suppose that $f: A \to B$ is surjective and $g \circ f = I_A$. Then claim (a) implies that $f^{-1} \subseteq g$; thus it suffices to show that $g \subseteq f^{-1}$. Let $(y, x) \in g$. Then by the surjectivity of f there exists $x_1 \in A$ such that $y = f(x_1)$ or equivalently, $(y, x_1) \in f^{-1}$. On the other hand,
$$x = g(y) = g(f(x_1)) = (g \circ f)(x_1) = I_A(x_1) = x_1 .$$
Therefore, $g \subseteq f^{-1}$.
3. Now suppose that $f: A \to B$ is injective and $f \circ g = I_B$. Then claim (b) implies that $g \subseteq f^{-1}$; thus it suffices to show that $f^{-1} \subseteq g$. Let $(y, x) \in f^{-1}$ or equivalently, $(x, y) \in f$ or $y = f(x)$. By the fact that $f \circ g = I_B$, we have $f(g(y)) = y$; thus the injectivity of f implies that $g(y) = x$ or $(y, x) \in g$. Therefore, $f^{-1} \subseteq g$ which completes the proof. □

§4.4 Inverse Functions

Since we have shown in the previous theorem that for functions $f: A \to B$ and $g: B \to A$,

1. $g = f^{-1}$ if and only if $g \circ f = I_A$ and $f \circ g = I_B$,
2. If f is surjective, and $g \circ f = I_A$, then $g = f^{-1}$,
3. If f is injective, and $f \circ g = I_B$, then $g = f^{-1}$,

we can conclude the following

Corollary

If $f: A \to B$ is an one-to-one correspondence, and $g: B \to A$ be a function. Then $g = f^{-1}$ if and only if $g \circ f = I_A$ or $f \circ g = I_B$.

Example

Let $A = \mathbb{R}$ and $B = \{x | x \geqslant 0\}$. Define $f: A \to B$ by $f(x) = x^2$ and $g: B \to A$ by $g(y) = \sqrt{y}$. Then $f \circ g = I_B$ but g is not inverse function of f since $(g \circ f)(x) = |x|$ for all $x \in A$.

§4.4 Inverse Functions

Definition

Let A be a non-empty set. A ***permutation*** of A is a one-to-one correspondence from A onto A.

Theorem

Let A be a non-empty set. Then

1. *the identity map I_A is a permutation of A.*
2. *the composite of permutations of A is a permutation of A.*
3. *the inverse of a permutation of A is a permutation of A.*
4. *if f is a permutation of A, then $f \circ I_A = I_A \circ f = f$.*
5. *if f is a permutation of A, then $f \circ f^{-1} = f^{-1} \circ f = I_A$.*
6. *if f and g are permutations of A, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

§4.5 Set Images

Definition

Let $f: A \to B$ be a function, and $X \subseteq A$, $Y \subseteq B$. The ***image*** of X (under f) or ***image set*** of X, denoted by $f(X)$, is the set

$$f(X) = \{y \in B \mid y = f(x) \text{ for some } x \in X\} = \{f(x) \mid x \in X\},$$

and the ***pre-image*** of Y (under f) or the ***inverse image*** of Y, denoted by $f^{-1}(Y)$, is the set

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

Remark: Here are some facts about images of sets that follow from the definitions:

(a) If $a \in D$, then $f(a) \in f(D)$.

(b) If $a \in f^{-1}(E)$, then $f(a) \in E$.

(c) If $f(a) \in E$, then $a \in f^{-1}(E)$.

(d) If $f(a) \in f(D)$ and f is one-to-one, then $a \in D$.