§4.2 Construction of Functions

Theorem

Suppose that f and g are functions. Then $f \cap g$ is a function with domain $A = \{x \mid f(x) = g(x)\}$, and $f \cap g = f|_A = g|_A$.

Proof.

Let $(x, y) \in f \cap g$. Then $y = f(x) = g(x)$; thus

$$\mathrm{Dom}(f \cap g) = \{x \mid f(x) = g(x)\} (\equiv A) .$$

If $(x, y_1), (x, y_2) \in f \cap g$, $(x, y_1), (x, y_2) \in f$ which, by the fact that f is a function, implies that $y_1 = y_2$. Therefore, $f \cap g$ is a function. Moreover,

$$f \cap g = \{(x, y) \mid \exists x \in A, y = f(x)\}$$

which implies that $f \cap g = f|_A$. □

§4.2 Construction of Functions

For $f \cup g$ being a function, it is (sufficient and) necessary that if $x \in \text{Dom}(f) \cap \text{Dom}(g)$, then $f(x) = g(x)$. Moreover, if $f \cup g$ is a function, then $f = (f \cup g)|_{\text{Dom}(f)}$ and $g = (f \cup g)|_{\text{Dom}(g)}$. In particular, we have the following

Theorem

Let f and g be functions with $\mathrm{Dom}(f) = A$ and $\mathrm{Dom}(g) = B$. If $A \cap B = \varnothing$, then $f \cup g$ is a function with domain $A \cup B$. Moreover,

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ g(x) & \text{if } x \in B\,. \end{cases}$$

§4.2 Construction of Functions

Theorem

Let f and g be functions with $\mathrm{Dom}(f) = A$ and $\mathrm{Dom}(g) = B$. If $A \cap B = \varnothing$, then $f \cup g$ is a function with domain $A \cup B$. Moreover,

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A\,, \\ g(x) & \text{if } x \in B\,. \end{cases} \qquad (\star)$$

Proof.

Clearly $\mathrm{Dom}(f \cup g) = A \cup B$. Suppose that $(x, y_1), (x, y_2) \in f \cup g$. If $(x, y_1) \in f$, then $x \in \mathrm{Dom}(f)$; thus by the fact that $A \cap B = \varnothing$, we must have $(x, y_2) \in f$. Since f is a function, $y_1 = f(x) = y_2$. Similarly, if $(x, y_1) \in g$, then $(x, y_2) \in g$ which also implies that $y_1 = g(x) = y_2$. Therefore, $f \cup g$ is a function and $(\star)$ is valid. □

§4.2 Construction of Functions

Definition

Let f be a real-valued function defined on an interval $I \subseteq \mathbb{R}$.

1. The function f is said to be ***increasing*** / ***decreasing*** on I if $x \leqslant y$ implies that $f(x) \leqslant f(y)$ / $f(x) \geqslant f(y)$ for all $x, y \in I$.
2. The function f is said to be ***strictly increasing*** / ***strictly decreasing*** on I if $x < y$ implies that $f(x) < f(y)$ / $f(x) > f(y)$ for all $x, y \in I$.

§4.3 Functions that are Onto; One-to-One Functions

Definition

Let $f\colon A \to B$ be a function.

1. The function f is said to be ***surjective*** or ***onto*** B if $\mathrm{Rng}(f) = B$. When f is surjective, f is called a surjection, and we write $f\colon A \xrightarrow{\text{onto}} B$.
2. The function f is said to be ***injective*** or ***one-to-one*** if it holds that "$f(x) = f(y) \Rightarrow x = y$". When f is injective, f is called a injection, and we write $f\colon A \xrightarrow{1-1} B$.
3. The function f is called a ***bijection*** if it is both injective and surjective. When f is a bijection, we write $f\colon A \xrightarrow[\text{onto}]{1-1} B$.

§4.3 Functions that are Onto; One-to-One Functions

Remark:

1. It is always true that $\text{Rng}(f) \subseteq B$; thus $f : A \to B$ is onto if and only if $B \subseteq \text{Rng}(f)$. In other words, $f : A \to B$ is onto if and only if every $b \in B$ has a pre-image. Therefore, to prove that $f : A \to B$ is onto B, it is sufficient to show that for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.
2. The direct proof of that $f : A \to B$ is injective is to verify the property that "$f(x) = f(y) \Rightarrow x = y$". A proof of the injectivity of f by contraposition assumes that $x \neq y$ and one needs to show that $f(x) \neq f(y)$.

§4.3 Functions that are Onto; One-to-One Functions

Theorem

1. *If* $f : A \to B$ *is onto* B *and* $g : B \to C$ *is onto* C*, then* $g \circ f$ *is onto* C*.*
2. *If* $f : A \to B$ *is one-to-one and* $g : B \to C$ *is one-to-one, then* $g \circ f$ *is one-to-one.*

Proof.

1. Let $c \in C$. By the surjectivity of g, there exists $b \in B$ such that $g(b) = c$. The surjectivity of f then implies the existence of $a \in A$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = g(f(a)) = g(b) = c$ which concludes ①.
2. Assume that $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$; thus by the injectivity of g, $f(x) = f(y)$. Therefore, the injectivity of f implies that $x = y$ which concludes ②. □

§4.3 Functions that are Onto; One-to-One Functions

Theorem

If $f : A \to B$, $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Theorem

Let $f : A \to B$ and $g : B \to C$ be functions.

1. *If $g \circ f$ is onto C, then g is onto C.*
2. *If $g \circ f$ is one-to-one, then f is one-to-one.*

Proof.

1. Let $c \in C$. Since $g \circ f$ is onto C, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a)$. Then $g(b) = g(f(a)) = (g \circ f)(a) = c$.
2. Suppose that $f(x) = f(y)$. Then $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$, and the injectivity of $g \circ f$ implies that $x = y$.

□