分析導論 II MA2050-*

Chapter 10. Applications

§10.1 Application on Signal Processing

In the study of **signal processing**, the Fourier transform and the inverse Fourier transform are often defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx\,,\ \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi}\,d\xi \quad \forall\, f \in L^1(\mathbb{R}^n)\,.$$

For $T \in \mathcal{S}(\mathbb{R}^n)'$, the Fourier transform of T is defined again by

$$\langle \widehat{T}, \phi\rangle = \langle T, \widehat{\phi}\rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^n)\,.$$

We also note that the definitions of the translation, dilation, and reflection of tempered distributions are **independent** of the Fourier transform, and are still defined by

$$\langle \tau_h T, \phi\rangle = \langle T, \tau_{-h}\phi\rangle\,,\ \langle d_\lambda T, \phi\rangle = \langle T, \lambda^n d_{\lambda^{-1}}\phi\rangle\,,\ \langle \widetilde{T}, \phi\rangle = \langle T, \widetilde{\phi}\rangle$$

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Concerning the convolution, we consider the $*$ convolution operator

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \quad \forall\, f, g \in L^1(\mathbb{R}^n).$$

instead of $\star$ convolution operators (which is $*/\sqrt{2\pi}^{\,n}$). The convolution of T and $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\langle T * f, \phi \rangle = \langle T, \widetilde{f} * \phi \rangle = \langle \widetilde{T}, f * \widetilde{\phi} \rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^n)\,.$$

Then for all $T \in \mathcal{S}(\mathbb{R}^n)'$,

1. $\check{\hat{T}} = \hat{\check{T}} = T$;
2. $\widehat{\tau_h T}(\xi) = \hat{T}(\xi)e^{-2\pi i \xi\cdot h}$, $\widehat{d_\lambda T} = \lambda^n d_{\frac{1}{\lambda}} \hat{T}$, $\hat{\hat{T}} = \check{T}$.
3. $\widehat{T * f} = \hat{T}\hat{f}$ and $\widehat{fT} = \hat{f} * \hat{T}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, if $S \in \mathcal{S}(\mathbb{R}^n)'$ has the property that $S * \phi \in \mathcal{S}(\mathbb{R}^n)$ for all $\phi \in \mathbb{R}^n$, then $\widehat{T * S} = \hat{T}\hat{S}$ in $\mathcal{S}(\mathbb{R}^n)'$ for all $T \in \mathcal{S}(\mathbb{R}^n)'$.

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$$\langle T*f,\phi\rangle=\langle T,\tilde{f}*\phi\rangle=\langle \tilde{T},f*\tilde{\phi}\rangle \qquad \forall\,\phi\in\mathcal{S}(\mathbb{R}^n)\,.$$

Then for all $T\in\mathcal{S}(\mathbb{R}^n)'$,

1. $\check{\hat{T}}=\hat{\check{T}}=T$;
2. $\widehat{\tau_h T}(\xi)=\hat{T}(\xi)e^{-2\pi i\xi\cdot h}$, $\widehat{d_\lambda T}=\lambda^n d_{\frac{1}{\lambda}}\hat{T}$, $\hat{\hat{T}}=\check{T}$.
3. $\widehat{T*f}=\hat{T}\hat{f}$ and $\widehat{fT}=\hat{f}*\hat{T}$ for all $f\in\mathcal{S}(\mathbb{R}^n)$. Moreover, if $S\in\mathcal{S}(\mathbb{R}^n)'$ has the property that $S*\phi\in\mathcal{S}(\mathbb{R}^n)$ for all $\phi\in\mathbb{R}^n$, then $\widehat{T*S}=\hat{T}\hat{S}$ in $\mathcal{S}(\mathbb{R}^n)'$ for all $T\in\mathcal{S}(\mathbb{R}^n)'$.

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Moreover,

1. $\widehat{\delta} = \check{\delta} = 1$ in $\mathcal{S}(\mathbb{R}^n)'$, and $\widehat{\delta_h}(\xi) = \widehat{\tau_h\delta}(\xi) = \widecheck{\delta_{-h}} = \widecheck{\tau_{-h}\delta} = e^{-2\pi i h\cdot\xi}$ in $\mathcal{S}(\mathbb{R}^n)'$ for all $h \in \mathbb{R}^n$.
2. By Euler's identity,
$$\widehat{\cos(2\pi\omega x)}(\xi) = \frac{1}{2}(\delta_\omega + \delta_{-\omega}), \quad \widehat{\sin(2\pi\omega x)}(\xi) = \frac{1}{2i}(\delta_\omega - \delta_{-\omega}).$$
3. $\delta * \delta = \delta$, and $\delta_a * \delta_b = \delta_{a+b}$ for all $a, b \in \mathbb{R}^n$.
4. $\delta * \phi = \phi$ and $\delta_a * \phi = \tau_a\phi$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.
5. Re-define the rect function $\Pi : \mathbb{R} \to \mathbb{R}$ by
$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < 1/2, \\ 0 & \text{if } |x| \geqslant 1/2. \end{cases}$$
Then $\widehat{\Pi}(\xi) = \check{\Pi}(\xi) = \mathrm{sinc}(\xi)$, where sinc is the **normalized sinc function**.

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6. Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be the triangle function define by

$$\Lambda(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

Then by the fact that Λ is an even function, if $\xi \neq 0$,

$$\begin{aligned} \widehat{\Lambda}(\xi) &= 2\int_0^1 (1-x)\cos(2\pi x\xi)\,dx \\ &= 2\Big[(1-x)\frac{\sin(2\pi x\xi)}{2\pi\xi}\Big|_{x=0}^{x=1} + \int_0^1 \frac{\sin(2\pi x\xi)}{2\pi\xi}\,dx\Big] \\ &= \frac{1-\cos(2\pi\xi)}{2\pi^2\xi^2} = \frac{\sin^2\pi\xi}{\pi^2\xi^2}\,, \end{aligned}$$

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When a continuous function, $x(t)$, is sampled at a constant rate F_s samples per second（以每秒 F_s 次均勻取樣）, there is always an unlimited number of other continuous functions that fit the same set of samples; however, only one of them is bandlimited to $F_s/2$ cycles per second (hertz), which means that its Fourier transform, $\widehat{x}(f)$, is 0 for all $|f| \geqslant F_s/2$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is said to be a ***bandlimited*** function if $\text{spt}(\widehat{f})$ is bounded. The ***bandwidth*** of a bandlimited function f is the number $\sup \text{spt}(\widehat{f})$. f is said to be ***timelimited*** if $\text{spt}(f)$ is bounded.

Recall that **the support of a function is the closure of the set on which the function has non-zero value.**

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Definition

In signal processing, the ***Nyquist rate*** is twice the bandwidth of a bandlimited function or a bandlimited channel.

In the field of **digital** signal processing, the ***sampling theorem*** is a fundamental bridge between continuous-time signals (often called "analog signals") and discrete-time signals (often called "digital signals"). It establishes a sufficient condition for a *sample rate*（取樣頻率）that permits a discrete sequence of samples to capture **all** the information from a continuous-time signal of finite bandwidth. To be more precise, Shannon's version of the theorem states that "if an analog signal contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart."

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In the following, we examine the sampling theorem rigorously. We start with the simplest version that the signal is continuous and integrable.

Theorem

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous integrable function. If $\text{spt}(\widehat{f})\subseteq[-B,B]$, then f is fully determined by the sequence $\Big\{f\Big(\frac{k}{2B}\Big)\Big\}_{k=-\infty}^{\infty}$, and

$$f(x)=\sum_{k=-\infty}^{\infty}f\Big(\frac{k}{2B}\Big)\mathrm{sinc}(2Bx-k)\qquad\forall\,x\in\mathbb{R}\,. \tag{1}$$

Remark: Equation (1) is called the ***Whittaker–Shannon interpolation formula***.

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Remark: Suppose that $f \in \mathcal{C}(\mathbb{R};\mathbb{R})$ be such that $\widehat{f}$, in the sense of tempered distribution, belongs to $L^2(\mathbb{R})$ and has support in $[-B, B]$. By the definition of the Fourier transform for $\mathcal{S}(\mathbb{R})'$ we have

$$\langle \check{\widehat{f}} - f, \phi \rangle = 0 \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})\,;$$

thus by the fact that $\check{\widehat{f}} \in \mathcal{C}_b(\mathbb{R};\mathbb{R})$,

$$f(x) = \check{\widehat{f}}(x) = \int_{-B}^{B} \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, the Fourier coefficients of $\widehat{f}$ is again $\Big\{ \frac{1}{2B} f\Big(\frac{-k}{2B}\Big) \Big\}_{k=-\infty}^{\infty}$ so that the same argument of showing Shannon's Sampling Theorem establishes the Whittaker–Shannon interpolation formula.

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Remark (Inner product point of view): Let

$$e_k(x) = \mathrm{sinc}(x-k) = (\tau_k \mathrm{sinc})(x)\,.$$

Then $e_k \in L^2(\mathbb{R})$ since

$$\int_{\mathbb{R}} \big|e_k(x)\big|^2 dx = \int_{\mathbb{R}} \mathrm{sinc}^2 x dx = \int_{\mathbb{R}} \frac{\sin^2 \pi x}{\pi^2 x^2} dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx < \infty\,.$$

By the Plancherel formula (for L^2-functions),

$$\begin{aligned}\langle e_k, e_\ell\rangle_{L^2(\mathbb{R})} &= \langle \widecheck{\tau_k \mathrm{sinc}}, \widecheck{\tau_\ell \mathrm{sinc}}\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Pi(\xi) e^{2\pi i k\xi}\,\overline{\Pi(\xi) e^{2\pi i \ell\xi}}\, d\xi \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(k-\ell)\xi}\, d\xi\end{aligned}$$

which is 0 if $k \neq \ell$ and is 1 is $k = \ell$. Therefore, we find that $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$.

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$$\langle e_k, e_\ell \rangle_{L^2(\mathbb{R})} = \big\langle \widehat{\tau_k \mathrm{sinc}}, \widehat{\tau_\ell \mathrm{sinc}} \big\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Pi(\xi) e^{2\pi i k \xi} \overline{\Pi(\xi) e^{2\pi i \ell \xi}} \, d\xi$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (k-\ell)\xi} \, d\xi$$

which is 0 if $k \neq \ell$ and is 1 is $k = \ell$. Therefore, we find that $\{e_k\}_{k \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$.

§10.1 Application on Signal Processing

Remark (Inner product point of view): Let

$$e_k(x) = \mathrm{sinc}(x-k) = (\tau_k \mathrm{sinc})(x)\,.$$

Then $e_k \in L^2(\mathbb{R})$ since

$$\int_{\mathbb{R}} \big|e_k(x)\big|^2 dx = \int_{\mathbb{R}} \mathrm{sinc}^2 x dx = \int_{\mathbb{R}} \frac{\sin^2 \pi x}{\pi^2 x^2} dx = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx < \infty\,.$$

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Now suppose that $f \in L^2(\mathbb{R})$ (so that $\widehat{f} \in L^2(\mathbb{R})$ by the Plancherel formula) such that $\text{spt}(\widehat{f}) \subseteq (-1/2, 1/2)$. Then

$$\langle f, e_k \rangle_{L^2(\mathbb{R})} = \langle \widehat{f}, \widehat{\tau_k \operatorname{sinc}} \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \widehat{f}(\xi) \overline{\Pi(-\xi) e^{-2\pi i k \xi}} \, d\xi$$
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if f is continuous at k. By the previous remark, if $f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ such that $\text{spt}(\widehat{f}) \subseteq (-1/2, 1/2)$, then

$$f(x) = \sum_{k=-\infty}^{\infty} f(k) \operatorname{sinc}(x-k) = \sum_{k=-\infty}^{\infty} (f, e_k)_{L^2(\mathbb{R})} e_k(x) \quad \forall\, x \in \mathbb{R}\,.$$

In other words, one can treat $\{e_k\}_{k \in \mathbb{Z}}$ as an "orthonormal basis" in the space

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§10.1 Application on Signal Processing

Lemma (Poisson summation formula)

Let the Fourier transform and the inverse Fourier transform be defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx \,, \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi \quad \forall\, f \in L^1(\mathbb{R}^n)\,.$$

Then

$$\sum_{n=-\infty}^{\infty} \phi(x+n) = \sum_{k=-\infty}^{\infty} \widehat{\phi}(k) e^{2\pi i k x} \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})\,.$$

The convergences on both sides are uniform.

Remark: Using the original definition of the Fourier transform, for $\phi \in \mathcal{S}(\mathbb{R})$ one has

$$\sum_{n=-\infty}^{\infty} \phi(x+2n\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) e^{inx}\,.$$

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§10.1 Application on Signal Processing

The **Shah** function, also called the **(Dirac) Comb** function and is denoted by Ш, is a tempered distribution defined by

$$\langle \text{Ш}, \phi \rangle = \sum_{n=-\infty}^{\infty} \phi(n) \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})\,.$$

We note that the sum above makes sense if $\phi \in \mathcal{S}(\mathbb{R})$, and

$$\sum_{n=-\infty}^{\infty} \phi(n) = \sum_{n=-\infty}^{\infty} \langle n \rangle^{-k} \langle n \rangle^{k} \phi(n) \leqslant \Big(\sum_{n=-\infty}^{\infty} \langle n \rangle^{-k} \Big) p_k(\phi) = C_k p_k(\phi)$$

for all $k \geqslant 2$. Therefore, Ш is indeed a tempered distribution. Since $\phi(n) = \langle \delta_n, \phi \rangle$, symbolically we also write $\text{Ш} = \sum_{n=-\infty}^{\infty} \delta_n$.

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§10.1 Application on Signal Processing

- **Properties of the Shah function**:

1. By the definition of the Fourier transform of tempered distributions,

$$\langle \widehat{\text{Ш}}, \phi \rangle = \langle \text{Ш}, \widehat{\phi} \rangle = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})\,,$$

and the Poisson summation formula implies that

$$\langle \widehat{\text{Ш}}, \phi \rangle = \sum_{k=-\infty}^{\infty} \phi(k) = \langle \text{Ш}, \phi \rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})\,.$$

Therefore, $\widehat{\text{Ш}} = \check{\text{Ш}} = \text{Ш}$ in $\mathcal{S}(\mathbb{R})'$.

§10.1 Application on Signal Processing

2. For $p \neq 0$, define $\text{Ш}_p = \frac{1}{p} d_p \text{Ш}$, where d_p is a dilation operator. Then using

$$\langle d_\lambda T, \phi \rangle = \langle T, \lambda^n d_{\lambda^{-1}} \phi \rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^n)\,,$$

we find that for $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle \text{Ш}_p, \phi \rangle = \Big\langle \frac{1}{p} d_p \text{Ш}, \phi \Big\rangle = \langle \text{Ш}, d_{p^{-1}} \phi \rangle = \sum_{n=-\infty}^{\infty} (d_{p^{-1}} \phi)(n)$$

$$= \sum_{n=-\infty}^{\infty} \phi(pn) = \sum_{n=-\infty}^{\infty} \langle \delta_{pn}, \phi \rangle .$$

In symbol, $\text{Ш}_p = \sum\limits_{n=-\infty}^{\infty} \delta_{pn}$. Moreover,

$$\widehat{\text{Ш}_p} = \widecheck{\text{Ш}_p} = d_{p^{-1}} \text{Ш} = \frac{1}{p} \text{Ш}_{\frac{1}{p}} .$$

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§10.1 Application on Signal Processing

3. For $f:\mathbb{R}\to\mathbb{R}$, let $f\,Ш_p:\mathcal{S}(\mathbb{R})\to\mathbb{C}$ be defined by

$$\langle f\,Ш_p,\phi\rangle=\sum_{n=-\infty}^{\infty}f(pn)\phi(pn)\qquad\forall\,\phi\in\mathcal{S}(\mathbb{R})\,.$$

Then $f\,Ш_p\in\mathcal{S}(\mathbb{R})'$ provided that $\sum\limits_{n=-\infty}^{\infty}\langle n\rangle^{-k}|f(pn)|<\infty$ for some $k\in\mathbb{N}\cup\{0\}$ since

$$\begin{aligned}\big|\langle f\,Ш_p,\phi\rangle\big| &\leqslant \sum_{n=-\infty}^{\infty}\big|f(pn)\big|\langle pn\rangle^{-k}\langle pn\rangle^{k}\big|\phi(pn)\big| \\ &\leqslant \max\{1,p^{-k}\}\Big(\sum_{n=-\infty}^{\infty}\langle n\rangle^{-k}\big|f(pn)\big|\Big)p_k(\phi)\,.\end{aligned}$$

In particular, $f\,Ш_p\in\mathcal{S}(\mathbb{R})'$ if $f\in\mathcal{S}(\mathbb{R})$. Moreover, we have

$$\widehat{Ш_p*\phi}=\widehat{\phi}\,\widehat{Ш_p}=\frac{1}{p}\widehat{\phi}\,Ш_{\frac{1}{p}}\qquad\forall\,\phi\in\mathcal{S}(\mathbb{R})\,.$$

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4. Suppose that $f\,\text{Ш}_p \in \mathcal{S}(\mathbb{R})'$. If $\phi, \psi \in \mathcal{S}(\mathbb{R})$,

$$\langle f\,\text{Ш}_p * \phi, \psi\rangle = \langle f\,\text{Ш}_p, \widetilde{\phi} * \psi\rangle = \sum_{n=-\infty}^{\infty} f(pn)(\widetilde{\phi} * \psi)(pn)$$

$$= \sum_{n=-\infty}^{\infty} f(pn) \int_{\mathbb{R}} \phi(y-pn)\psi(y)\,dy$$

$$= \sum_{n=-\infty}^{\infty} f(pn)\langle \tau_{pn}\phi, \psi\rangle .$$

which shows that (in symbol)

$$f\,\text{Ш}_p * \phi = \sum_{n=-\infty}^{\infty} f(pn)\tau_{pn}\phi \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R})$$

whenever $f\,\text{Ш}_p \in \mathcal{S}(\mathbb{R})'$.

§10.1 Application on Signal Processing

Definition

Let $T \in \mathcal{S}(\mathbb{R}^n)'$ be a tempered distribution. The support of T, denoted by $\mathrm{spt}(T)$, is the complement of the open set $O = \bigcup_{U \in \mathcal{F}(T)} U$, where $\mathcal{F}(T)$ is a collection of open sets given by

$$\mathcal{F}(T) = \Big\{ U \subseteq \mathbb{R}^n \text{ open} \,\Big|\, \phi \in \mathcal{S}(\mathbb{R}^n) \wedge \mathrm{spt}(\phi) \subseteq U \Rightarrow \langle T, \phi \rangle = 0 \Big\} .$$

The definition above implies that if U is open and $U \subseteq O = \mathrm{spt}(T)^{\complement}$, then $U \in \mathcal{F}(T)$ (since if $U \notin \mathcal{F}(T)$, then any open set containing U does not belong to $\mathcal{F}(T)$ which results in that $O \cap U = \varnothing$); thus $\mathrm{spt}(T)^{\complement}$ is the "largest" open set in $\mathcal{F}(T)$. Moreover, the support of a tempered distribution must be closed; thus if $\mathrm{spt}(\widehat{f}) \subseteq (-B, B)$, there exists $0 < R < B$ such that $\mathrm{spt}(\widehat{f}) \subseteq [-R, R]$. In particular, a choice of R is the supremum of $\mathrm{spt}(\widehat{f})$, the bandwidth of f.

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§10.1 Application on Signal Processing

Example

Let $\omega \in \mathbb{R}^n$. The support of δ_ω, the delta function at ω, is $\{\omega\}$. To see this, let U be an open set in $\mathbb{R}^n$ and $\omega \notin U$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\text{spt}(\phi) \subseteq U$, then $\langle \delta_\omega, \phi \rangle = \phi(\omega) = 0$; thus every open set U that does not contain x belongs to $\mathcal{F}(\delta_\omega)$. This implies that $\bigcup_{U \in \mathcal{F}(\delta_\omega)} U = \mathbb{R}^n \backslash \{x\}$; thus the support of δ_ω is $\{\omega\}$.

Example

Let $T \in \mathcal{S}(\mathbb{R})$ be the tempered distribution $T = \delta_\omega - \delta_{-\omega}$, where $\omega \neq 0$. Then $\text{spt}(T) = \{\omega, -\omega\}$. Since the Fourier transform of the signal $f(t) = \sin(2\pi\omega t)$ is $\widehat{f}(\xi) = \dfrac{\delta_\omega - \delta_{-\omega}}{2i}$, we find that $\text{spt}(\widehat{f}) = \{-\omega, \omega\}$.

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§10.1 Application on Signal Processing

Lemma

Let $T \in \mathcal{S}(\mathbb{R}^n)'$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. If $spt(T) \cap spt(\phi) = \varnothing$, then $\langle T, \phi \rangle = 0$.

Proof.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{spt}(T) \cap \operatorname{spt}(\phi) = \varnothing$, and $O = \operatorname{spt}(T)^\complement$. Then $\operatorname{spt}(\phi) \subseteq \operatorname{spt}(T)^\complement = O$. Since $O \in \mathcal{F}(T)$, $\langle T, \phi \rangle = 0$. □

Theorem (Not known if it is true)

Let $f \in \mathcal{C}(\mathbb{R};\mathbb{R}) \cap \mathcal{S}(\mathbb{R})'$. If $spt(\widehat{f}) \subseteq (-B, B)$, then f is fully determined by the sequence $\left\{ f\left(\frac{k}{2B}\right) \right\}_{k=-\infty}^{\infty}$, and

$$f = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2B}\right) \tau_{\frac{k}{2B}} d_{\frac{1}{2B}} \mathrm{sinc} \qquad in \quad \mathcal{S}(\mathbb{R})'.$$

§10.1 Application on Signal Processing

Lemma

Let $T \in \mathcal{S}(\mathbb{R}^n)'$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. If spt$(T) \cap$ spt$(\phi) = \varnothing$, then $\langle T, \phi \rangle = 0$.

Proof.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that spt$(T) \cap$ spt$(\phi) = \varnothing$, and $O =$ spt$(T)^{\complement}$. Then spt$(\phi) \subseteq$ spt$(T)^{\complement} = O$. Since $O \in \mathcal{F}(T)$, $\langle T, \phi \rangle = 0$. □

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§10.1 Application on Signal Processing

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Let $T \in \mathcal{S}(\mathbb{R}^n)'$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$. *If* $\mathrm{spt}(T) \cap \mathrm{spt}(\phi) = \varnothing$, *then* $\langle T, \phi \rangle = 0$.

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Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathrm{spt}(T) \cap \mathrm{spt}(\phi) = \varnothing$, and $O = \mathrm{spt}(T)^\complement$. Then $\mathrm{spt}(\phi) \subseteq \mathrm{spt}(T)^\complement = O$. Since $O \in \mathcal{F}(T)$, $\langle T, \phi \rangle = 0$. □

Theorem (Not known if it is true)

Let $f \in \mathcal{C}(\mathbb{R};\mathbb{R}) \cap \mathcal{S}(\mathbb{R})'$. *If* $\mathrm{spt}(\widehat{f}) \subseteq (-B, B)$, *then* f *is fully determined by the sequence* $\Big\{ f\big(\frac{k}{2B}\big) \Big\}_{k=-\infty}^{\infty}$, *and*

$$f = \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{2B}\Big) \tau_{\frac{k}{2B}} d_{\frac{1}{2B}} \operatorname{sinc} \qquad \textit{in} \quad \mathcal{S}(\mathbb{R})' .$$

§10.2 Application on Partial Differential Equations

Recall that in mathematical modelling we talked about the heat equation

$$u_t - \Delta u = f \quad \text{in} \quad \Omega \times (0, T)\,, \tag{2a}$$

$$u = u_0 \quad \text{on} \quad \Omega \times \{t = 0\}\,, \tag{2b}$$

together with one of the following boundary condition (called (2c)):

1. **Dirichlet B.C.**: $u = g$ on $\partial\Omega$.
2. **Neumann B.C.**: $\dfrac{\partial u}{\partial \mathbf{N}} = g$ on $\partial\Omega$, where $\dfrac{\partial u}{\partial \mathbf{N}} = \nabla u \cdot \mathbf{N}$ is the directional derivative of u in the outward-pointing direction $\mathbf{N}$.
3. **Robin B.C.**: $\dfrac{\partial u}{\partial \mathbf{N}} + u = g$ on $\partial\Omega$.
4. **Periodic B.C.**: $u(0, t) = u(L, t)$ for all $t > 0$.

Here $\Omega \subseteq \mathbb{R}^n$ is an open set, and the functions f, g and h are given.

§10.2 Application on Partial Differential Equations

§10.2.1 The case $\Omega = (0, L)$

- **Dirichlet B.C.**: Here the Dirichet boundary condition becomes $u(0,t) = a(t)$ and $u(L,t) = b(t)$ for some given functions a, b.

Let $v(x,t) = u(x,t) - \dfrac{b(t) - a(t)}{L}x - a(t)$. Then v satisfies

$$v_t - v_{xx} = F \quad \text{in} \quad (0,L) \times (0,T)\,, \tag{3a}$$

$$v = v_0 \quad \text{on} \quad (0,L) \times \{t = 0\}\,, \tag{3b}$$

$$v = 0 \quad \text{on} \quad \{0,L\} \times (0,T)\,, \tag{3c}$$

where $F(x,t) = f(x,t) - \dfrac{b'(t) - a'(t)}{L}x - a'(t)$, and $v_0(x) = u_0(x) - \dfrac{b(0) - a(0)}{L}x - a(0)$. In other words, W.L.O.G. we can assume that $a = b = 0$.

§10.2 Application on Partial Differential Equations

Now consider

$$\begin{aligned} u_t - u_{xx} &= f && \text{in} \quad (0, L) \times (0, T)\,, \\ u &= u_0 && \text{on} \quad (0, L) \times \{t = 0\}\,, \\ u &= 0 && \text{on} \quad \{0, L\} \times (0, T)\,. \end{aligned}$$

The idea of solving the PDE above is to express the solution $u(x, t)$, for each $t \in (0, T)$ as a Fourier series. There are three possible choices:

1. $u(x, t) = \dfrac{c_0(t)}{2} + \sum\limits_{k=1}^{\infty} \Big[c_k(t) \cos \dfrac{2\pi kx}{L} + s_k(t) \sin \dfrac{2\pi kx}{L}\Big]$.
2. $u(x, t) = \dfrac{c_0(t)}{2} + \sum\limits_{k=1}^{\infty} c_k(t) \cos \dfrac{\pi kx}{L}$: the cosine series of u.
3. $u(x, t) = \sum\limits_{k=1}^{\infty} s_k(t) \sin \dfrac{\pi kx}{L}$: the sine series of u.

§10.2 Application on Partial Differential Equations

Due to the boundary condition, we choose the sine series to represent the solution. We also represent the initial data u_0 and the forcing f using the sine series

$$u_0(x) = \sum_{k=1}^{\infty} \widehat{u}_{0k} \sin \frac{\pi kx}{L}, \quad f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{\pi kx}{L},$$

and assume that

$$u_t(x,t) = \sum_{k=1}^{\infty} \frac{\partial}{\partial t}\Big[s_k(t) \sin \frac{\pi kx}{L}\Big], \quad u_{xx}(x,t) = \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}\Big[s_k(t) \sin \frac{\pi kx}{L}\Big].$$

Then $\{s_k(t)\}_{k=1}^{\infty}$ satisfies

$$\sum_{k=1}^{\infty} \Big(s_k'(t) + \frac{\pi^2 k^2}{L^2} s_k(t)\Big) \sin \frac{\pi kx}{L} = \sum_{k=1}^{\infty} f_k(t) \frac{\sin \pi kx}{L},$$

$$\sum_{k=1}^{\infty} s_k(0) \sin \frac{\pi kx}{L} = \sum_{k=1}^{\infty} \widehat{u}_{0k} \sin \frac{\pi kx}{L}.$$

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§10.2 Application on Partial Differential Equations

Therefore, for each $k \in \mathbb{N}$ the function $s_k(t)$ satisfies the IVP

$$s_k'(t) + \frac{\pi^2 k^2}{L^2} s_k(t) = f_k(t)\,, \qquad s_k(0) = \widehat{u_0}_k\,.$$

Method of Integrating factor:

Multiplying both sides by $Q_k(t) \equiv \exp\Big(\frac{\pi^2 k^2 t}{L^2}\Big)$,

$$\frac{d}{dt}\big[Q_k(t)s_k(t)\big] = Q_k(t) f_k(t)\,;$$

thus

$$Q_k(t)s_k(t) - s_k(0) = \int_0^t Q_k(s) f_k(s)\, ds\,.$$

Therefore, we expect that the solution is given by

$$u(x,t) = \sum_{k=1}^{\infty} \Big[e^{-\frac{\pi^2 k^2 t}{L^2}} \widehat{u_0}_k + \int_0^t e^{\frac{\pi^2 k^2 (s-t)}{L^2}} f_k(s)\, ds \Big] \sin\frac{\pi k x}{L}\,.$$

§10.2 Application on Partial Differential Equations

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§10.2 Application on Partial Differential Equations

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§10.2 Application on Partial Differential Equations

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Multiplying both sides by $Q_k(t) \equiv \exp\left(\frac{\pi^2 k^2 t}{L^2}\right)$

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§10.2 Application on Partial Differential Equations

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§10.2 Application on Partial Differential Equations

- **Neumann B.C.**: Here the Neumann boundary condition becomes $u_x(0,t) = a(t)$ and $u_x(L,t) = b(t)$ for some given functions a, b.

Let $v(x,t) = u(x,t) - \dfrac{b(t) - a(t)L}{2L}x^2 - a(t)x$. Then v satisfies

$$v_t - v_{xx} = F \quad \text{in} \quad (0, L) \times (0, T)\,, \tag{4a}$$

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$$v_x = 0 \quad \text{on} \quad \{0, L\} \times (0, T)\,, \tag{4c}$$

where $F(x,t) = f(x,t) + \dfrac{b(t) - a(t)L}{L} - a'(t)x$, and $v_0(x) = u_0(x) - \dfrac{b(0) - a(0)L}{2L}x^2 - a(0)x$. In other words, W.L.O.G. we can assume that $a = b = 0$.

§10.2 Application on Partial Differential Equations

Due to the boundary condition, we choose the cosine series to represent the solution:

$$u(x,t) = \frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t) \cos \frac{\pi k x}{L}.$$

We also represent the initial data u_0 and the forcing f using the sine series

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and assume that

$$u_t(x,t) = \sum_{k=1}^{\infty} \frac{\partial}{\partial t}\Big[c_k(t) \cos \frac{\pi k x}{L}\Big], \quad u_{xx}(x,t) = \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}\Big[c_k(t) \cos \frac{\pi k x}{L}\Big].$$

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§10.2 Application on Partial Differential Equations

Then $\{c_k(t)\}_{k=0}^{\infty}$ satisfies

$$\frac{c_0'(t)}{2} + \sum_{k=1}^{\infty} \Big(c_k'(t) + \frac{\pi^2 k^2}{L^2} c_k(t)\Big) \cos\frac{\pi kx}{L} = \frac{f_0(t)}{2} + \sum_{k=1}^{\infty} f_k(t) \frac{\cos \pi kx}{L},$$

$$\frac{c_0(0)}{2} + \sum_{k=1}^{\infty} c_k(0) \cos\frac{\pi kx}{L} = \frac{\widehat{u}_{00}}{2} + \sum_{k=1}^{\infty} \widehat{u}_{0k} \cos\frac{\pi kx}{L}.$$

The comparison of coefficients shows that c_k satisfies the IVP

$$c_0'(t) = f_0(t), \qquad c_0(0) = \widehat{u}_{00}$$

$$c_k'(t) + \frac{\pi^2 k^2}{L^2} c_k(t) = f_k(t), \qquad c_k(0) = \widehat{u}_{0k}.$$

and are given by

$$c_0(t) = \widehat{u}_{00} + \int_0^t f_0(s)\, ds, \quad c_k(t) = e^{-\frac{\pi^2 k^2 t}{L^2}} \widehat{u}_{0k} + \int_0^t e^{\frac{\pi^2 k^2 (s-t)}{L^2}} f_k(s)\, ds.$$

§10.2 Application on Partial Differential Equations

Then $\left\{c_k(t)\right\}_{k=0}^{\infty}$ satisfies

$$\frac{c_0'(t)}{2} + \sum_{k=1}^{\infty}\left(c_k'(t) + \frac{\pi^2 k^2}{L^2} c_k(t)\right)\cos\frac{\pi kx}{L} = \frac{f_0(t)}{2} + \sum_{k=1}^{\infty} f_k(t)\frac{\cos\pi kx}{L},$$
$$\frac{c_0(0)}{2} + \sum_{k=1}^{\infty} c_k(0)\cos\frac{\pi kx}{L} = \frac{\widehat{u}_{00}}{2} + \sum_{k=1}^{\infty}\widehat{u}_{0k}\cos\frac{\pi kx}{L}.$$

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§10.2 Application on Partial Differential Equations

Therefore, the solution to

$$\begin{aligned} u_t - u_{xx} = f \quad & \text{in} \quad (0, L) \times (0, T)\,, \\ u = u_0 \quad & \text{on} \quad (0, L) \times \{t = 0\}\,, \\ u_x = 0 \quad & \text{on} \quad \{0, L\} \times (0, T)\,, \end{aligned}$$

is

$$\begin{aligned} u(x, t) = & \frac{1}{2}\Big[\widehat{u}_{00} + \int_0^t f_0(s)\, ds\Big] \\ & + \sum_{k=1}^{\infty}\Big[e^{-\frac{\pi^2 k^2 t}{L^2}}\widehat{u}_{0k} + \int_0^t e^{\frac{\pi^2 k^2 (s-t)}{L^2}} f_k(s)\, ds\Big] \cos\frac{\pi k x}{L}\,. \end{aligned}$$