Exercise Problem Sets 12

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Problem 1. Show that the Fourier transform of a Schwartz function is also a Schwartz function.

Proof. Let $f \in \mathscr{S}(\mathbb{R}^n)$, $N \in \mathbb{N} \cup \{0\}$ be a given non-negative integer, and $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a given multi-index. The goal is to show that there exists a constant $C = C_{N,\alpha}$ such that

$$|\xi|^N \big| D^\alpha \widehat{f}(\xi) \big| \leqslant C_{N,\alpha} \qquad \forall\, \xi \in \mathbb{R}^n .$$

Note that $|\xi|^N \leqslant 1 + |\xi|^{2N}$ for all $\xi \in \mathbb{R}^n$; thus it suffices to show that there exists $C_{N,\alpha}$ such that

$$(1 + |\xi|^{2N}) \big| D^\alpha \widehat{f}(\xi) \big| \leqslant C_{N,\alpha} \qquad \forall\, \xi \in \mathbb{R}^n .$$

Using the notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for multi-index, by Corollary 9.10 and 9.12 in the lecture note we find that

$$(1 + |\xi|^{2N}) \big| D^\alpha \widehat{f}(\xi) \big| = (1 + |\xi|^{2N}) \Big| \mathscr{F}_x \big[x^\alpha f(x) \big] \Big| = \Big| \mathscr{F} \big[(1 + \Delta_x)^N \big[x^\alpha f(x) \big] \big](\xi) \Big| ,$$

where $\Delta_x = \sum\limits_{k=1}^n \dfrac{\partial^2}{\partial x_k^2} = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$. By Lipschitz rule, we find that

$$(1 + \Delta_x)^N \big[x^\alpha f(x) \big] = \sum_{|\beta|=0}^{2N} \mathcal{P}_{N,\alpha,\beta}(x) D^\beta f(x)$$

for some polynomials $\mathcal{P}_\beta$. For each multi-index β, by the fact that $D^\beta f \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{P}_\beta D^\beta f \in \mathscr{S}(\mathbb{R}^n)$; thus Proposition 9.4 and Lemma 9.8 imply that for each multi-index β there exists $C_{N,\alpha,\beta}$ such that

$$\Big| \mathscr{F}_x \big[\mathcal{P}_\beta(x) D^\beta f(x) \big](\xi) \Big| \leqslant C_{N,\alpha,\beta} \qquad \forall\, \xi \in \mathbb{R}^n .$$

Therefore, for all $\xi \in \mathbb{R}^n$,

$$|\xi|^N \big| D^\alpha \widehat{f}(\xi) \big| \leqslant \sum_{|\beta|=0}^{2N} \Big| \mathscr{F}_x \big[\mathcal{P}_\beta(x) D^\beta f(x) \big](\xi) \Big| \leqslant \sum_{|\beta|=0}^{2N} C_{N,\alpha,\beta} \equiv C_{N,\alpha} .$$

□

Problem 2. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is continuous, absolutely integrable, and $\widehat{f}(\xi) = \dfrac{\ln(1+\xi^2)}{\xi^2}$. Find $f(0)$ and $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$.

Solution. By the Fourier inversion formula,

$$\begin{aligned} f(0) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi\cdot 0}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\ln(1+\xi^2)}{\xi^2}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \Big[\frac{-\ln(1+\xi^2)}{\xi} \Big|_{\xi=-\infty}^{\xi=\infty} + \int_{\mathbb{R}} \frac{1}{\xi} \frac{2\xi}{1+\xi^2}\, d\xi \Big] \\ &= \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1+\xi^2}\, d\xi = \sqrt{\frac{2}{\pi}} \arctan \xi \Big|_{\xi=-\infty}^{\xi=\infty} = \sqrt{\frac{2}{\pi}} \cdot \pi = \sqrt{2\pi} . \end{aligned}$$

Moreover, by the definition and the property of the Fourier transform,

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{\xi \to 0} \sqrt{2\pi} \widehat{f}(\xi) = \sqrt{2\pi} \lim_{t \to 0^+} \frac{\ln(1+t)}{t} = \sqrt{2\pi} .$$

□

Problem 3. 1. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous integrable function such that $\widehat{f}$ is also integrable. Show that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) \cos[(x-y)\xi] \, dy \Big) \, d\xi \qquad \forall \, x \in \mathbb{R} \, .$$

2. If in addition to condition in 1, f is an even function. Show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty f(y) \cos(x\xi) \cos(y\xi) \, dy \Big) \, d\xi \, .$$

3. If in addition to condition in 1, f is an odd function. Show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty f(y) \sin(x\xi) \sin(y\xi) \, dy \Big) \, d\xi \, .$$

4. For a function $g : [0, \infty) \to \mathbb{C}$ satisfying $\displaystyle\int_0^\infty \big|g(x)\big| \, dx < \infty$, the Fourier cosine transform and the Fourier sine transform of g, denoted by $\mathscr{F}_{\cos}[g]$ and $\mathscr{F}_{\sin}[g]$ respectively, are functions defined by

$$\mathscr{F}_{\cos}[g](\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \cos(y\xi) \, dy \qquad \text{and} \qquad \mathscr{F}_{\sin}[g](\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \sin(y\xi) \, dy \, .$$

(a) Show that if $\mathscr{F}_{\cos}[g] \in L^1(\mathbb{R})$, then

$$g(x) = \mathscr{F}_{\cos}\big[\mathscr{F}_{\cos}[g]\big](x) \qquad \text{whenever } x \in [0, \infty) \text{ and } g \text{ is continuous at } x,$$

or equivalently,

$$g(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty g(y) \cos(y\xi) \, dy \Big) \cos(x\xi) \, d\xi$$

whenever $x \in [0, \infty)$ and g is continuous at x.

(b) Show that if $\mathscr{F}_{\sin}[g] \in L^1(\mathbb{R})$, then

$$g(x) = \mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) \qquad \text{whenever } x \in [0, \infty) \text{ and } g \text{ is continuous at } x,$$

or equivalently,

$$g(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty g(y) \sin(y\xi) \, dy \Big) \sin(x\xi) \, d\xi$$

whenever $x \in (0, \infty)$ and g is continuous at x.

Hint of 4: Consider the even or odd extension of g, and apply conclusions in 2 and 3.

Proof. 1. Let f be a continuous integrable function such that $\widehat{f}$ is also integrable. Then $\check{f}$ is also integrable; thus the Fourier inversion formula implies that

$$f(x) = \check{\hat{f}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{-iy\xi} \, dy \Big) e^{ix\xi} \, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{i(x-y)\xi} \, dy \Big) \, d\xi$$

and

$$f(x) = \hat{\check{f}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{iy\xi} \, dy \Big) e^{-ix\xi} \, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{-i(x-y)\xi} \, dy \Big) \, d\xi$$

whenever f is continuous at x. Therefore, if f is continuous at x, then

$$\begin{aligned}
f(x) &= \frac{1}{2}\Big[\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)e^{i(x-y)\xi}\,dy\Big)\,d\xi + \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)e^{-i(x-y)\xi}\,dy\Big)\,d\xi\Big] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\frac{e^{i(x-y)\xi}+e^{-i(x-y)\xi}}{2}\,dy\Big)\,d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\cos[(x-y)\xi]\,dy\Big)\,d\xi\,.
\end{aligned}$$

We note that by the sum and difference of angles identities, the identity above implies that

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\big[\cos(x\xi)\cos(y\xi)+\sin(x\xi)\sin(y\xi)\big]\,dy\Big)\,d\xi\,. \tag{0.1}$$

2. If f is an even function, then $\displaystyle\int_{\mathbb{R}} f(y)\sin(x\xi)\sin(y\xi)\,dy = 0$; thus (0.1) shows that if f is continuous at x,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\cos(x\xi)\cos(y\xi)\,dy\Big)\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(2\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi\,.$$

Note that the inner integral is an even function of ξ, so

$$f(x) = \frac{2}{2\pi}\int_0^\infty\Big(2\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi = \frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi\,.$$

3. If f is an odd function, then $\displaystyle\int_{\mathbb{R}} f(y)\cos(x\xi)\cos(y\xi)\,dy = 0$; thus (0.1) shows that if f is continuous at x,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\sin(x\xi)\sin(y\xi)\,dy\Big)\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(2\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi\,.$$

Note that the inner integral is an odd function of ξ, so

$$f(x) = \frac{2}{2\pi}\int_0^\infty\Big(2\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi = \frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi\,.$$

4. Suppose that $g : [0,\infty) \to \mathbb{C}$ is integrable.

   (a) Let $f : \mathbb{R} \to \mathbb{C}$ be defined by

$$f(x) = \begin{cases} g(x) & \text{if } x > 0\,, \\ -g(-x) & \text{if } x < 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

   Then f is an odd function and is integrable on $\mathbb{R}$. Moreover,

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-iy\xi}\,dy = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)\big[\cos(y\xi)-i\sin(y\xi)\big]\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)\cos(y\xi)\,dy - i\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)\sin(y\xi)\,dy\,.
\end{aligned}$$

By the definition of f,

$$\begin{aligned}\int_{\mathbb{R}} f(y)\cos(y\xi)\,dy &= \int_0^\infty f(y)\cos(y\xi)\,dy + \int_{-\infty}^0 f(y)\cos(y\xi)\,dy \\ &= \int_0^\infty g(y)\cos(y\xi)\,dy - \int_{-\infty}^0 g(-y)\cos(y\,xi)\,dy \\ &= \int_0^\infty g(y)\cos(y\xi)\,dy - \int_{\infty}^0 g(y)\cos(-y\xi)\,d(-y) = 0\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} f(y)\sin(y\xi)\,dy &= \int_0^\infty f(y)\sin(y\xi)\,dy + \int_{-\infty}^0 f(y)\sin(y\xi)\,dy \\ &= \int_0^\infty g(y)\sin(y\xi)\,dy - \int_{-\infty}^0 g(-y)\sin(y\,xi)\,dy \\ &= \int_0^\infty g(y)\sin(y\xi)\,dy - \int_{\infty}^0 g(y)\sin(-y\xi)\,d(-y) \\ &= 2\int_0^\infty g(y)\sin(y\xi)\,dy = \sqrt{2\pi}\mathscr{F}_{\sin}[g](\xi)\,;\end{aligned}$$

thus $\widehat{f} = -i\mathscr{F}_{\sin}[g]$ which implies that $\widehat{f} \in L^1(\mathbb{R})$. On the other hand, $\check{f}(\xi) = \widehat{f}(-\xi) = i\mathscr{F}_{\sin}[g](\xi)$; thus the Fourier inversion formula implies that

$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) = -i\mathscr{F}_{\sin}\big[i\mathscr{F}_{\sin}[g]\big](x) = \check{\widehat{f}}(x) = f(x)$$

whenever f is continuous at x. In particular, if $x \in (0,\infty)$ and g is continuous at x, then f is continuous at x and $f(x) = g(x)$ which imply that

$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) = g(x) \qquad \text{whenever } x \in (0,\infty) \text{ and } g \text{ is continuous at } x.$$ □

Problem 4. A vector-valued function $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$ is called a Schwartz function, still denoted by $\boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)$, if $u_j \in \mathscr{S}(\mathbb{R}^n)$ for all $1 \leqslant j \leqslant n$. Show the Korn inequality

$$\sum_{i,j=1}^n \big\|\epsilon_{ij}(\boldsymbol{u})\big\|_{L^2(\mathbb{R}^n)}^2 \geqslant \frac{1}{2}\sum_{i,j=1}^n \Big\|\frac{\partial u_j}{\partial x_i}\Big\|_{L^2(\mathbb{R}^n)}^2 \qquad \forall\, \boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)\,,$$

where $\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the symmetric part of $D\boldsymbol{u}$.

Hint: Use the Plancherel formula.

Proof. By the Plancherel formula,

$$\begin{aligned}\big\|\epsilon_{ij}(u)\big\|_{L^2(\mathbb{R}^n)}^2 &= \frac{1}{4}\sum_{i,j=1}^n \int_{\mathbb{R}^n} \big[\xi_i\xi_i\widehat{u}_j(\xi)\overline{\widehat{u}_j(\xi)} + \xi_j\xi_j\widehat{u}_i(\xi)\overline{\widehat{u}_i(\xi)} + \xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)} + +\xi_j\xi_i\overline{\widehat{u}_i(\xi)}\widehat{u}_j(\xi)\big]\,d\xi \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 + 2\xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)}\big]\,d\xi \\ &\geqslant \sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 - \xi_i^2|\widehat{u}_i(\xi)|^2 - \xi_j^2|\widehat{u}_j(\xi)|^2\big]\,d\xi \\ &\geqslant \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2\big]\,d\xi\end{aligned}$$

$$\geqslant \frac{1}{2} \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \xi_i^2 |\widehat{u}_j(\xi)|^2 d\xi = \frac{1}{2} \sum_{i,j=1}^{n} \left\| \frac{\partial u_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^n)}^2 . \qquad \square$$

Problem 5. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support; that is, $\{x \in \mathbb{R}^n \mid \phi(x) \neq 0\}$ is bounded. Show that if $f \in L^1_{\rm loc}(\mathbb{R}^n)$, then the convolution $\phi * f$ is smooth and

$$D^\alpha(\phi * f)(x) = \big[(D^\alpha \phi) * f\big](x) = \int_{\mathbb{R}^n} (D^\alpha \phi)(x-y) f(y)\, dy \,,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index.

Note that the standard mollifiers $\{\eta_\varepsilon\}_{\varepsilon>0}$ are one of such kind of functions, so this problem shows that $\eta_\varepsilon * f$ is smooth if $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

Proof. By Theorem 5.40 in the lecture note, it suffices to show that

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \phi_{x_j}(x-y) f(y)\, dy \qquad \forall\, x \in \mathbb{R}^n \tag{0.2}$$

and the right-hand side is a continuous function (in x). The continuity of the right-hand side function follows directly from the Dominated Convergence Theorem: If $\{x_k\}_{k=1}^\infty$ is a sequence with limit x (W.L.O.G. we can assume that $|x_k - x| < 1$ for all $k \in \mathbb{N}$) and ϕ is supported inside $B(0,R)$; that is, $\{z \in \mathbb{R}^n \mid \phi(z) \neq 0\} \subseteq B(0,R)$, then the fact that

$$\big|\phi_{x_j}(x_k - y) f(y)\big| \leqslant M \mathbf{1}_{B(0,R+|x|+1)} |f(y)| \quad \text{ whenever } \quad y \in \mathbb{R}^n$$

and the right-hand side functions is integrable on $\mathbb{R}^n$, the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \phi_{x_j}(x_k - y) f(y)\, dy = \int_{\mathbb{R}^n} \lim_{k\to\infty} \phi_{x_j}(x_k - y) f(y)\, dy = \int_{\mathbb{R}^n} \phi_{x_j}(x-y) f(y)\, dy$$

which shows that the right-hand side function in (0.2) is continuous (in x).

Let $x \in \mathbb{R}^n$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence converging to 0. W.L.O.G., we assume that $|h_k| < 1$ for all $k \in \mathbb{N}$. Define

$$g_k(y) = \frac{\phi(x + h_k \mathbf{e}_j - y) - \phi(x-y)}{h_k} f(y) \,.$$

By the fact that ϕ has compact support, $M \equiv \sup_{z\in\mathbb{R}^n} \big|\phi_{x_j}(z)\big| < \infty$. By the mean value theorem,

$$\Big|\frac{\phi(x + h_k \mathbf{e}_j - y) - \phi(x-y)}{h_k}\Big| \leqslant M \mathbf{1}_{B(x,R+1)}(y)$$

so that

$$\big|g_k(y)\big| \leqslant M \mathbf{1}_{B(x,R+1)}(y) \big|f(y)\big| \qquad \forall\, y \in \mathbb{R}^n \text{ and } k \in \mathbb{N} \,,$$

where again $R > 0$ is chosen so that ϕ is supported in $B(0,R)$. Since $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the function on the right-hand side of the inequality above is an integrable function. Therefore, the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} g_k(y)\, dy = \int_{\mathbb{R}^n} \lim_{k\to\infty} g_k(y)\, dy = \int_{\mathbb{R}^n} \phi_{x_j}(x-y) f(y)\, dy$$

which shows (0.2). $\square$

Problem 6. 1. Let d_r denote the dilation operator defined by $d_r f(x) = f\big(\frac{x}{r}\big)$. Show that

$$\mathscr{F}(d_r f) = r^n d_{1/r}\mathscr{F}(f) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{0.3}$$

2. In some occasions (especially in engineering applications), the Fourier transform and inverse Fourier transform of a (Schwartz) function f are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x\cdot\xi}dx \qquad \text{and} \qquad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{i2\pi x\cdot\xi}d\xi\,.$$

Show that under this definition, $\check{\widehat{f}} = \widehat{\check{f}} = f$ for all $f \in \mathscr{S}(\mathbb{R}^n)$. Note that you can use the Fourier Inversion Formula that we derive in class.

Proof. Let $\mathscr{F}$ denote the Fourier transform operator that we used in class, and $\widehat{\ }$ be the Fourier transform operator in this problem.

1. Let d_r denote the dilation operator define by $(d_r f)(x) = f(rx)$. By the change of variables formula,

$$\begin{aligned}
\mathscr{F}(d_r f)(\xi) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}(d_r f)(x)e^{-ix\cdot\xi}\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(r^{-1}x)e^{-ix\cdot\xi}\,dx \\
&= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y)e^{-iry\cdot\xi}r^n\,dy = \frac{r^n}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y)e^{-iy\cdot(r\xi)}\,dy \\
&= r^n\mathscr{F}(f)(r\xi) = r^n\big[d_{\frac{1}{r}}\mathscr{F}(f)\big](\xi)
\end{aligned}$$

so that (0.3) is established.

2. Replacing f by $d_{1/r}f$ in (0.3) implies that

$$\mathscr{F}(f) = \mathscr{F}\big(d_r d_{\frac{1}{r}} f\big) = r^n d_{\frac{1}{r}}\mathscr{F}\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond$}$$

Similarly, $\mathscr{F}^*(d_r f) = r^n d_{\frac{1}{r}}\mathscr{F}^*(f)$ so that

$$\mathscr{F}^*(f) = r^n d_{\frac{1}{r}}\mathscr{F}^*\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond\diamond$}$$

Note that

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx = \sqrt{2\pi}^n\mathscr{F}(f)(2\pi\xi) = \sqrt{2\pi}^n\big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi) \\
&= \frac{1}{\sqrt{2\pi}^n}(2\pi)^n\big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi}f)(\xi)
\end{aligned}$$

and

$$\check{f}(\xi) = \widehat{f}(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi}f)(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi}f)(\xi)\,.$$

Therefore, $(\diamond)$ implies that

$$\begin{aligned}
\check{\widehat{f}}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi}\widehat{f})(\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*\Big(\frac{1}{\sqrt{2\pi}^n}d_{2\pi}\mathscr{F}(d_{2\pi}f)\Big)(\xi) \\
&= \mathscr{F}^*\big((2\pi)^{-n}d_{2\pi}\mathscr{F}(d_{2\pi}f)\big)(\xi) = \mathscr{F}^*(\mathscr{F}f)(\xi) = f(\xi)\,.
\end{aligned}$$

Similarly, $(\diamond\diamond)$ implies that

$$\widehat{\check{f}}(\xi) = \mathscr{F}\big((2\pi)^{-n}d_{2\pi}\mathscr{F}^*(d_{2\pi}f)\big)(\xi) = \mathscr{F}(\mathscr{F}^*f)(\xi) = f(\xi)\,.$$

□