Exercise Problem Sets 9

Apr. 14. 2023

Problem 1. Let $f \in \mathscr{C}(\mathbb{T})$ and $\{\widehat{f}_k\}_{k=-\infty}^{\infty}$ be the Fourier coefficients defined by $\widehat{f}_k = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$. Show that if $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| < \infty$, then the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof. Let $M_k = |\widehat{f}_k|$ and $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| = M$. Then $\big|s_n(f,x)\big| \leqslant M$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Moreover,

$$\big|\widehat{f}_k e^{ikx}\big| \leqslant M_k \quad \forall\, x \in \mathbb{R} \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty} M_k = M < \infty\,.$$

Therefore, the Weierstrass M-test implies that the Fourier series converges uniformly on $\mathbb{R}$. Suppose that the Fourier series converges uniformly to g. Then $\big|g(x)\big| \leqslant M$ for all $x \in \mathbb{R}$; thus Problem 9 in Exercise 4 implies that the Cesàro mean of $\{s_k(f,\cdot)\}_{k=1}^{\infty}$ converges uniformly to g on $\mathbb{R}$. Since $f \in \mathscr{C}(\mathbb{T})$, the Cesàro mean of the Fourier series of f converges uniformly to f on $\mathbb{R}$; thus $f = g$. □

Problem 2. Compute the Fourier series of the function $f : (-\pi,\pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & -\pi < x < 0\,, \\ \pi - x & 0 \leqslant x < \pi\,, \end{cases}$$

and show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,. \tag{0.1}$$

Also use the Fourier series of the function $y = x^2$

$$s(x^2,x) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos kx$$

to conclude (0.1).

Solution. We compute the Fourier coefficients as follows. For $k \in \mathbb{N}$,

$$s_k = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(kx)\,dx = \frac{1}{\pi}\Big[\frac{-(\pi-x)\cos(kx)}{k}\Big|_{x=0}^{x=\pi} - \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\Big] = \frac{1}{k}$$

and

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(kx)\,dx = \frac{1}{\pi}\Big[\frac{(\pi-x)\sin(kx)}{k}\Big|_{x=0}^{x=\pi} + \frac{1}{k}\int_0^{\pi}\sin(kx)\,dx\Big] \\ &= \frac{-\cos(kx)}{k^2\pi}\Big|_{x=0}^{x=\pi} = \frac{1-(-1)^k}{k^2\pi}\,, \end{aligned}$$

while

$$c_0 = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{\pi}{2}\,.$$

Therefore, by the fact that $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = \pi$,

$$\frac{\pi}{4} + \sum_{k=1}^{\infty}\Big(\frac{1-(-1)^k}{k^2\pi}\cos(kx) + \frac{1}{k}\sin(kx)\Big) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0\,, \\ \pi - x & \text{if } 0 < x \leqslant \pi\,, \\ \dfrac{\pi}{2} & \text{if } x = 0\,. \end{cases}$$

We note that the case $x = 0$ implies that

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{1-(-1)^k}{k^2\pi}$$

which shows the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}\,.$$

We also note that the identity above can be obtained by

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$$

so that

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}\,.$$ □

Problem 3. The proof of Theorem 8.25 in the lecture note only establishes the validity of the theorem for the case $L = \pi$. Use this fact to show that the conclusion also holds for general $L > 0$.

Proof. Suppose that the theorem holds for the case $L = \pi$. Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f\big(\frac{Lx}{\pi}\big)$ (or equivalently, $f(x) = g\big(\frac{\pi x}{L}\big)$). Then g is 2π-periodic piecewise Hölder continuous exponent $\alpha \in (0, 1]$, and

$$s_n(g, x) = s_n\big(f, \frac{Lx}{\pi}\big) \qquad \text{and} \qquad s_n(f, x) = s_n\big(g, \frac{\pi x}{L}\big)\,.$$

Therefore, by the fact that $\lim_{x\to x_0^\pm} h(cx) = \lim_{y\to(cx_0)^\pm} h(x)$ if $c > 0$,

$$\begin{aligned}\lim_{n\to\infty} s_n(f, x_0) &= \lim_{n\to\infty} s_n\big(g, \frac{\pi x_0}{L}\big) = \frac{1}{2}\Big[\lim_{y\to(\frac{\pi x_0}{L})^+} g(y) + \lim_{y\to(\frac{\pi x_0}{L})^-} g(y)\Big] \\ &= \frac{1}{2}\Big[\lim_{x\to x_0^+} g\big(\frac{\pi x}{L}\big) + \lim_{x\to x_0^-} g\big(\frac{\pi x}{L}\big)\Big] = \frac{1}{2}\Big[\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\Big] \\ &= \frac{f(x_0^+) + f(x_0^-)}{2}\,.\end{aligned}$$

Moreover, if x_0 is a jump discontinuity of f, then $\dfrac{\pi x_0}{L}$ is a jump discontinuity of g so that

$$\begin{aligned}\lim_{n\to\infty} s_n\big(f, x_0 + \frac{L}{n}\big) &= \lim_{n\to\infty} s_n\Big(g, \frac{\pi}{L}\big(x_0 + \frac{L}{n}\big)\Big) = \lim_{n\to\infty} s_n\big(g, \frac{\pi x_0}{L} + \frac{\pi}{n}\big) \\ &= \lim_{y\to(\frac{\pi x_0}{L})^+} g(y) + c\Big[\lim_{y\to(\frac{\pi x_0}{L})^+} g(y) - \lim_{y\to(\frac{\pi x_0}{L})^-} g(y)\Big] \\ &= \lim_{x\to x_0^+} g\big(\frac{\pi x}{L}\big) + c\Big[\lim_{x\to x_0^+} g\big(\frac{\pi x}{L}\big) - \lim_{x\to x_0^-} g\big(\frac{\pi x}{L}\big)\Big] = f(x_0^+) + ca\,.\end{aligned}$$

Similarly, $\lim_{n\to\infty} s_n\big(f, x_0 + \frac{L}{n}\big) = f(x_0^-) - ca.$ $\square$

Problem 4. For a given function $f : [0, L] \to \mathbb{R}$, the even extension of f is a function $\bar{f} : [-L, L] \to \mathbb{R}$ such that

$$\bar{f}(x) = f(x) \text{ if } x \in [0, L) \qquad \text{and} \qquad \bar{f}(x) = f(-x) \text{if } x \in [-L, 0)\,.$$

1. Let $f : [0, L] \to \mathbb{R}$ be an integrable function. The cosine series of f is the Fourier series of the even extension of f. Find the cosine series of f.

2. Suppose in addition $f : [0, L] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Show that the cosine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$.

Proof. 1. Let $\bar{f}$ be the even extension of f, and $\{c_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of $\bar{f}$. Then by the fact that $\bar{f}$ is even, $s_k = 0$ for all $k \in \mathbb{N}$. Moreover,

$$\begin{aligned} c_k &= \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\cos\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx + \frac{1}{L}\int_{-L}^{0} f(-x)\cos\frac{k\pi x}{L}\,dx \\ &= \frac{1}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx + \frac{1}{L}\int_L^0 f(x)\cos\frac{k\pi(-x)}{L} d(-x) \\ &= \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx\,. \end{aligned}$$

Therefore, the cosine series of f is

$$s(\bar{f}, x) = \frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\cos\frac{k\pi y}{L}\,dy\Big)\cos\frac{k\pi x}{L}\,.$$

2. If f is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$, then the odd extension $\bar{f}$ of f is also piecewise Hölder continuous with exponent $\alpha \in (0, 1]$; thus

$$s(\bar{f}, x_0) = \frac{\bar{f}(x_0^+) + \bar{f}(x_0^-)}{2} = \frac{f(x_0^+) + f(x_0^-)}{2}$$

which shows that the cosine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$. $\square$

Problem 5. For a given function $f : [0, L] \to \mathbb{R}$, the odd extension of f is a function $\bar{f} : [-L, L] \to \mathbb{R}$ such that

$$\bar{f}(x) = f(x) \text{ if } x \in [0, L) \qquad \text{and} \qquad \bar{f}(x) = -f(-x) \text{if } x \in [-L, 0)\,.$$

1. Let $f : [0, L] \to \mathbb{R}$ be an integrable function. The sine series of f is the Fourier series of the odd extension of f. Find the sine series of f.

2. Suppose in addition $f : [0, L] \to \mathbb{R}$ is piecewise Hölder continuous with exponent $\alpha \in (0, 1]$. Show that the sine series of f at $x_0 \in (0, L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$.

Proof. 1. Let $\bar{f}$ be the odd extension of f, and $\{c_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of $\bar{f}$. Then by the fact that $\bar{f}$ is odd, $c_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Moreover,

$$\begin{aligned} s_k &= \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\sin\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx - \frac{1}{L}\int_{-L}^0 f(-x)\sin\frac{k\pi x}{L}\,dx \\ &= \frac{1}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx - \frac{1}{L}\int_L^0 f(x)\sin\frac{k\pi(-x)}{L}d(-x) \\ &= \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\,. \end{aligned}$$

Therefore, the sine series of f is

$$s(\bar{f},x) = \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\sin\frac{k\pi y}{L}\,dy\Big)\sin\frac{k\pi x}{L}\,.$$

2. If f is piecewise Hölder continuous with exponent $\alpha \in (0,1]$, then the odd extension $\bar{f}$ of f is also piecewise Hölder continuous with exponent $\alpha \in (0,1]$; thus

$$s(\bar{f},x_0) = \frac{\bar{f}(x_0^+) + \bar{f}(x_0^-)}{2} = \frac{f(x_0^+) + f(x_0^-)}{2}$$

which shows that the sine series of f at $x_0 \in (0,L)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$. □

Problem 6. Let f be the sinc function defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Show that

$$f(x) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_n\cos(nx)\,, \qquad \forall\, x \in [-\pi,\pi]\,,$$

where $b_n = \dfrac{1}{\pi}\displaystyle\int_{(n-1)\pi}^{(n+1)\pi}\frac{\sin x}{x}\,dx$. Use this result to compute $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\,dx$.

Proof. Since f is an even function, the Fourier coefficients $\{s_k\}_{k=1}^{\infty}$ is the zero sequence; that is,

$$s_n(f,x) = \frac{c_0}{2} + \sum_{k=1}^{n} c_k\cos kx\,,$$

where

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin x}{x}\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{2\sin x\cos kx}{x}\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{\sin(k+1)x - \sin(k-1)x}{x}\,dx \\ &= \frac{1}{\pi}\Big[\int_0^{\pi}\frac{\sin(k+1)}{x}\,dx - \int_0^{\pi}\frac{\sin(k-1)x}{x}\,dx\Big] = \frac{1}{\pi}\Big[\int_0^{(k+1)\pi}\frac{\sin y}{y}\,dy - \int_0^{(k-1)\pi}\frac{\sin y}{y}\,y \\ &= \frac{1}{\pi}\int_{(k-1)\pi}^{(k+1)\pi}\frac{\sin x}{x}\,dx \equiv b_k\,. \end{aligned}$$

Since $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$ and $\lim\limits_{x\to 0} f'(x) = 0$, we find that f' is bounded; thus f is Lipschitz continuous. Therefore, the Fourier series of f converges uniformly to f on $\mathbb{T}$; thus

$$f(x) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx \qquad \forall\, x \in \mathbb{R}$$

and the convergence is uniform. In particular, at $x = 0$ and $x = \pi$,

$$1 = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \qquad \text{and} \qquad 0 = \frac{b_0}{2} + \sum_{k=1}^{\infty} (-1)^k b_k\,.$$

Therefore,

$$\frac{1}{2} = \sum_{k=0}^{\infty} b_{2k+1} = \frac{1}{\pi} \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+2)\pi} \frac{\sin x}{x}\, dx = \frac{1}{\pi} \int_0^{\infty} \frac{\sin x}{x}\, dx$$

which shows that $\displaystyle\int_{-\pi}^{\pi} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. □