Exercise Problem Sets 8

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Problem 1. Let f be a 2π-periodic Lipchitz function. Show that for $n \geqslant 2$,

$$\|f - F_{n-1} \star f\|_\infty \leqslant \frac{1 + 2\log n}{2n}\pi \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \tag{0.1}$$

and

$$\big\|f - s_n(f,\cdot)\big\|_\infty \leqslant \frac{2\pi(1+\log n)^2}{n}\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}\,. \tag{0.2}$$

Inequality (0.2) provides the rate of convergence of the Fourier series to Lipschitz functions. What is the rate of convergence if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$?

Hint: For (0.1), apply the estimate

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}$$

in the following inequality:

$$\big|f(x) - F_{n-1} \star f(x)\big| \leqslant \Big[\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big]\big|f(x) - f(x-y)\big|F_{n-1}(y)\,dy$$

with $\delta = \dfrac{\pi}{n}$. For (0.2), use (8.2.7) in the lecture note and note that

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant \|f - F_n \star f\|_\infty\,.$$

Proof. Recall that the Fejér kernel F_n is given by

$$F_n(x) = \begin{cases} \dfrac{1}{2\pi(n+1)}\dfrac{\sin^2\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}} & \text{if } x \notin \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,, \\ \dfrac{n+1}{2\pi} & \text{if } x \in \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,. \end{cases}$$

Therefore, by the fact that $\sin|x| \geqslant \dfrac{2}{\pi}|x|$ for $|x| < \dfrac{\pi}{2}$, we find that

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}.$$

By the fact that $\displaystyle\int_{-\pi}^{\pi} F_{n-1}(x)\,dx = 0$ for all $n \geqslant 2$, we find that if $n \geqslant 2$ and $0 < \delta < \pi$,

$$\begin{aligned}
\big|f(x) - F_{n-1} \star f(x)\big| &= \Big|\int_{-\pi}^{\pi} f(x)F_{n-1}(x-y)\,dy - \int_{-\pi}^{\pi} f(y)F_{n-1}(x-y)\,dy\Big| \\
&= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(y)\big]F_{n-1}(x-y)\,dy\Big| \\
&= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(x-y)\big]F_{n-1}(y)\,dy\Big| \\
&= \Big|\Big(\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big)\big[f(x) - f(x-y)\big]F_{n-1}(y)\,dy\Big|\,.
\end{aligned}$$

Let $\delta = \dfrac{\pi}{n}$. Then

$$\left|\int_{-\delta}^{\delta} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{-\delta}^{\delta} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}|y| \cdot \frac{n}{2\pi}\,dy = \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{\pi}\int_0^{\delta} y\,dy$$
$$= \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2\pi}\frac{\pi^2}{n^2} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2n}\,.$$

Moreover,

$$\left|\int_{\delta\leqslant|y|\leqslant\pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{\delta\leqslant|y|\leqslant\pi} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}|y| \cdot \frac{\pi}{2ny^2}\,dy$$
$$= \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n}\int_{\delta}^{\pi}\frac{1}{y}\,dy = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n}\log\frac{\pi}{\delta} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}\log n}{n}\,.$$

The two inequalities above implies (0.1).

For the validity of (0.2), by the fact that

$$\inf_{p\in\mathscr{P}_n(\mathbb{T})}\|f-p\|_\infty \leqslant \|f - F_n \star f\|_\infty$$

we conclude from (8.2.7) in the lecture note and (0.1) that

$$\big\|f - s_n(f,\cdot)\big\|_\infty \leqslant (3+\log n)\|f - F_n\star f\|_\infty \leqslant \frac{(3+\log n)(1+2\log(n+1))}{2(n+1)}\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}$$

and the desired inequality follows from the fact that

$$\frac{(3+\log n)(1+2\log(n+1))}{2(n+1)} \leqslant \frac{2(1+\log n)^2}{n} \qquad \forall\, n\geqslant 2\,.$$

If $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $0<\alpha<1$, then

$$\left|\int_{-\delta}^{\delta} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{-\delta}^{\delta} \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}|y|^\alpha \cdot \frac{n}{2\pi}\,dy = \frac{n\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{\pi}\int_0^{\delta} y^\alpha\,dy$$
$$= \frac{n\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{(1+\alpha)\pi}\frac{\pi^{1+\alpha}}{n^{1+\alpha}} = \frac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{(1+\alpha)n^\alpha}$$

and

$$\left|\int_{\delta\leqslant|y|\leqslant\pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\,dy\right| \leqslant \int_{\delta\leqslant|y|\leqslant\pi} \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}|y|^\alpha \cdot \frac{\pi}{2ny^2}\,dy$$
$$= \frac{\pi\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{n}\int_{\delta}^{\pi}\frac{1}{y^{2-\alpha}}\,dy = \frac{\pi\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{n}\frac{\pi^{\alpha-1}-\delta^{\alpha-1}}{\alpha-1} \leqslant \frac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{(1-\alpha)n^\alpha}\,.$$

Therefore,

$$\big|f(x) - F_{n-1}\star f(x)\big| \leqslant \pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\Big(\frac{1}{1+\alpha}+\frac{1}{1-\alpha}\Big)\frac{1}{n^\alpha} = \frac{2\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{(1-\alpha^2)n^\alpha}\,.$$

The estimate above, together with (8.2.7) in the lecture note, shows that

$$\big|f(x) - s_n(f,x)\big| \leqslant \frac{2(3+\log n)\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{(1-\alpha^2)n^\alpha}\,.$$

□

Problem 2. In this problem, we are concerned with the following

Theorem 0.1 (Bernstein)**.** *Suppose that f is a 2π-periodic function such that for some constant C and $\alpha \in (0,1)$,*

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant Cn^{-\alpha}$$

for all $n \in \mathbb{N}$. Then $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$.

Complete the following to prove the theorem.

1. Show that

$$\|p'\|_\infty \leqslant n\|p\|_\infty \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T}). \tag{0.3}$$

2. Choose $p_n \in \mathscr{P}_n(\mathbb{T})$ such that $\|f - p_n\|_\infty \leqslant 2Cn^{-\alpha}$ for $n \in \mathbb{N}$. Define $q_0 = p_1$, and $q_n = p_{2^n} - p_{2^{n-1}}$ for $n \in \mathbb{N}$.

   (a) Show that $\sum\limits_{n=0}^{\infty} q_n = f$ and the convergence is uniform.

   (b) Show that

$$\big|q_n(x) - q_n(y)\big| \leqslant 6Cn2^{n(1-\alpha)}|x-y| \quad \text{and} \quad \big|q_n(x) - q_n(y)\big| \leqslant 12C2^{-n\alpha}.$$

   (c) For any $x, y \in \mathbb{T}$ with $|x - y| \leqslant 1$, choose $m \in \mathbb{N}$ such that $2^{-m} \leqslant |x-y| \leqslant 2^{1-m}$. Then use the inequality

$$\big|f(x) - f(y)\big| \leqslant \sum_{n=0}^{m-1} \big|q_n(x) - q_n(y)\big| + \sum_{n=m}^{\infty} \big|q_n(x) - q_n(y)\big|$$

   to show that $\big|f(x) - f(y)\big| \leqslant B|x-y|^\alpha$ for some constant $B > 0$.

Hint of 1: Suppose the contrary that there exists $p \in \mathscr{P}_n(\mathbb{T})$ such that $\|p'\|_\infty > n\|p\|_\infty$. By rescaling p and relabeling points in $\mathbb{T}$ if necessary, without loss of generality we can assume that

$$\|p'\|_\infty > n\,, \quad \|p\|_\infty < 1\,, \quad \text{and} \quad p'(0) = \|p'\|_\infty\,.$$

Choose $\gamma \in \big[-\frac{\pi}{2n}, \frac{\pi}{2n}\big]$ such that $\sin(n\gamma) = -p(0)$, and define $r(x) = \sin n(x - \gamma) - p(x)$. Show that r has at least $2n + 2$ distinct zeros in $[\alpha_{-n}, \alpha_n] \equiv \big[-\pi + \gamma + \frac{\pi}{2n}, \pi + \gamma + \frac{\pi}{2n}\big]$ by showing that r has at least one zero in (α_k, α_{k+1}), where $\alpha_k = \gamma + \frac{\pi}{n}\big(k + \frac{1}{2}\big)$ for each $|k| \leqslant n$, while r has at least 3 distinct zeros in (α_s, α_{s+1}) if $0 \in (\alpha_s, \alpha_{s+1})$ (in fact, $s = -1$). On the other hand, the fact that $r \in \mathscr{P}_n(\mathbb{T})$ implies that r has at most $2n$ distinct zeros in $\mathbb{T}$ unless r is the zero function which leads to a contradiction.

Proof. 1. Suppose the contrary that there is a trigonometric polynomial p of degree n such that

$$\|p'\|_\infty > n\|p\|_\infty\,.$$

By rescaling p and relabeling points in $\mathbb{T}$ if necessary, without loss of generality we can assume that

$$\|p'\|_\infty > n\,, \quad \|p\|_\infty < 1\,, \quad \text{and} \quad p'(0) = \|p'\|_\infty\,.$$

Choose $\gamma \in \big[-\frac{\pi}{2n}, \frac{\pi}{2n}\big]$ such that $\sin(n\gamma) = -p(0)$ (and $\cos(n\gamma) > 0$), and define

$$r(x) = \sin n(x-\gamma) - p(x)\,.$$

Then $r \in \mathscr{P}_n(\mathbb{T})$. Let $\alpha_k = \gamma + \frac{\pi}{n}\big(k+\frac{1}{2}\big)$. Note that $\alpha_k \neq 0$ for all $k \in \mathbb{Z}$ since if $\alpha_k = 0$ for some k, then by the fact that $\|p\|_\infty < 1$,

$$\sin n\gamma = -\sin\big[\pi\big(k+\frac{1}{2}\big)\big] = (-1)^{k+1} \neq -p(0)\,,$$

a contradiction.

Note that $\{\alpha_k\}_{k=-n}^{n}$ forms a partition of $[\alpha_{-n}, \alpha_n] \equiv \big[-\pi+\gamma+\frac{\pi}{2n}, \pi+\gamma+\frac{\pi}{2n}\big]$ which is an interval of length 2π. Since $|p(\alpha_k)| < 1$ and $r(\alpha_k) = (-1)^k - p(\alpha_k)$, the sign of $r(\alpha_k)$ is $(-1)^k$. By the intermediate value theorem, for each $|k| \leqslant n$ there exists $\beta_k \in (\alpha_k, \alpha_{k+1})$ such that $r(\beta_k) = 0$. Moreover, since $\gamma \in \big[-\frac{\pi}{2n}, \frac{\pi}{2n}\big]$, we must have $0 \in (\alpha_{-1}, \alpha_0)$; thus $\sin n(x-\gamma)$ increases from $-1 = \sin n(\alpha_{-1}-\gamma)$ to $1 = \sin n(\alpha_0-\gamma)$ in (α_{-1}, α_0), $r(\alpha_{-1}) < 0 < r(\alpha_0)$. Moreover, $r(0) = 0$, and $r'(0) = n\cos n\gamma - p'(0) < 0$. Therefore, there exist two small positive numbers ϵ_1 and ϵ_2 such that $r(-\epsilon_1) > 0$ and $r(\epsilon_2) < 0$. As a consequence, r has at least 3 zeros in (α_{-1}, α_0). This shows that r has at least $2n+2$ zeros in $\big[-\pi+\gamma+\frac{\pi}{2n}, \pi+\gamma+\frac{\pi}{2n}\big]$.

Now, by the fact that $r \in \mathscr{P}_n(\mathbb{T})$, r has at most $2n$ distinct zeros in (α_{-n}, α_n) unless r is a zero function. Therefore, $r \equiv 0$ which implies that

$$p(x) = \sin n(x-\gamma) \qquad \forall\, x \in \mathbb{T}\,.$$

The identity above then contradicts to the assumption that $p'(0) = \|p'\|_\infty > n$.

2. Choose $p_n \in \mathscr{P}_n(\mathbb{T})$ such that $\|f - p_n\|_\infty \leqslant 2Cn^{-\alpha}$ for $n \in \mathbb{N}$. We remark here that this implies that $\in L^\infty(\mathbb{T})$; that is, $\sup\limits_{x\in\mathbb{T}} \big|f(x)\big| < \infty$. Define $q_0 = p_1$, and $q_n = p_{2^n} - p_{2^{n-1}}$ for $n \in \mathbb{N}$. Note that

$$\sum_{n=0}^{\infty} q_n(x) = \lim_{n\to\infty} p_{2^n}(x) = f(x) \tag{$\diamond$}$$

and for $n \in \mathbb{N}$,

$$\|q_n\|_\infty \leqslant \|p_{2^n} - f\|_\infty + \|f - p_{2^{n-1}}\|_\infty \leqslant 2C2^{-n\alpha} + 2C2^{-(n-1)\alpha} \leqslant 6C2^{-n\alpha}$$

so that the Weierstrass M-test implies that the convergence in ($\diamond$) is uniform. Moreover, the mean value theorem and (0.3) imply that

$$\big|q_n(x) - q_n(y)\big| \leqslant \|q_n'\|_\infty |x-y| \leqslant 2^n \|q_n\|_\infty |x-y| \leqslant 6Cn2^{n(1-\alpha)}|x-y|.$$

On the other hand, we also have

$$\big|q_n(x)-q_n(y)\big| \leqslant 2\|q_n\|_\infty \leqslant 12C2^{-n\alpha}.$$

Therefore, for all $m \in \mathbb{N}$,

$$\begin{aligned}\big|f(x)-f(y)\big| &\leqslant \sum_{n=0}^{m-1}\big|q_n(x)-q_n(y)\big| + \sum_{n=m}^{\infty}\big|q_n(x)-q_n(y)\big| \\ &\leqslant \sum_{n=0}^{m-1} 6C2^{n(1-\alpha)}|x-y| + 12C\sum_{n=m}^{\infty}2^{-n\alpha} \\ &= 6C|x-y|\frac{2^{m(1-\alpha)}-1}{2^{1-\alpha}-1} + 12C\frac{2^{-m\alpha}}{1-2^{-\alpha}}\,.\end{aligned}$$

For fixed $x, y \in \mathbb{T}$ with $|x-y| \leqslant 1$, choose $m \in \mathbb{N}$ such that $2^{-m} \leqslant |x-y| \leqslant 2^{-m+1}$. Then

$$\begin{aligned}\big|f(x)-f(y)\big| &\leqslant 6C2^{-m+1}\frac{2^{m(1-\alpha)}-1}{2^{1-\alpha}-1} + 12C\frac{2^{-m\alpha}}{1-2^{-\alpha}} \\ &\leqslant 12C2^{-m\alpha}\Big[\frac{1}{2^{1-\alpha}-1}+\frac{1}{1-2^{-\alpha}}\Big] \leqslant B|x-y|^\alpha\,,\end{aligned}$$

where $B = 12C\Big[\dfrac{1}{2^{1-\alpha}-1}+\dfrac{1}{1-2^{-\alpha}}\Big]$; hence $\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \leqslant B + 3\|f\|_\infty$. □