Exercise Problem Sets 15

Dec. 31. 2022

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, and Df is a constant map in $\mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$; that is, $(Df)(x)(u) = (Df)(y)(u)$ for all $x, y \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Show that f is a linear term plus a constant and that the linear part of f is the constant value of Df.

Proof. Suppose that $(Df)(x) = L \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$, where L is a "constant" bounded linear map independent of x. Let $g(x) = f(x) - Lx$. Then $(Dg)(x) = (Df)(x) - L = 0$ for all $x \in \mathbb{R}^n$; thus Problem 2 of Exercise 15 implies that g is a constant function. Therefore,

$$f(x) - Lx = C$$

for some constant C which shows that $f(x) = Lx + C$; that is, f is a linear term plus a constant. ▫

Problem 2. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be of class $\mathscr{C}^k$ and $(D^\ell f)(a) = 0$ for $\ell = 1, \cdots, k-1$. Show that if $(D^k f)(a)(u, u, \cdots, u) > 0$ for all non-zero vectors $u \in \mathbb{R}^n$, then f has a local minimum at a; that is, there exists $\delta > 0$ such that

$$f(x) \geqslant f(a) \qquad \forall\, x \in B(a, \delta)\,.$$

Proof. Let $a \in U$. Since U is open, there exists $r > 0$ such that $B(a, r) \subseteq U$. Note that $g : B(a, r) \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x, u) = (D^k f)(x)(u, \cdots, u)$ is continuous since

$$\begin{aligned}
\big|g(x,u) - g(y,v)\big| &= \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(y)(v,\cdots,v)\big| \\
&\leqslant \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \Big|\big[(D^k f)(x) - (D^k f)(y)\big](v,\cdots,v)\Big| \\
&\leqslant \big|(D^k f)(x)(u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|^k_{\mathbb{R}^n} \\
&\leqslant \big|(D^k f)(x)(u-v,u,\cdots,u)\big| + \big|(D^k f)(x)(v,u,\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| \\
&\quad + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|^k_{\mathbb{R}^n} \\
&\leqslant \big\|(D^k f)(x)\big\|\|u-v\|_{\mathbb{R}^n}\|u\|^{k-1}_{\mathbb{R}^n} + \big|(D^k f)(x)(v,u-v,u\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| \\
&\quad + \big|(D^k f)(x)(v,v,u\cdots,u) - (D^k f)(x)(v,\cdots,v)\big| + \big|(D^k f)(x) - (D^k f)(y)\big\|\|v\|^k_{\mathbb{R}^n} \\
&\leqslant \cdots\cdots \\
&\leqslant \big\|(D^k f)(x)\big\|\|u-v\|_{\mathbb{R}^n}\big(\|u\|^{k-1}_{\mathbb{R}^n} + \|u\|^{k-2}_{\mathbb{R}^n}\|v\|_{\mathbb{R}^n} + \cdots + \|u\|_{\mathbb{R}^n}\|v\|^{k-2}_{\mathbb{R}^n} + \|v\|^{k-1}_{\mathbb{R}^n}\big) \\
&\quad + \big\|(D^k f)(x) - (D^k f)(y)\big\|\|v\|^k_{\mathbb{R}^n}
\end{aligned}$$

so that

$$\begin{aligned}
&\big|g(x,u) - g(y,v)\big| \\
&\qquad \leqslant \big\|(D^k f)(x)\big\|\big(\|u\|_{\mathbb{R}^n} + \|v\|_{\mathbb{R}^n}\big)^{k-1}\|u-v\|_{\mathbb{R}^n} + \big\|(D^k f)(x) - (D^k f)(y)\big\|\|v\|^k_{\mathbb{R}^n}
\end{aligned} \tag{0.1}$$

and the right-hand side approaches zero as $x \to y$ and $u \to v$. In particular, by the compactness of $\mathbb{S}^{n-1} \equiv \big\{x \in \mathbb{R}^n \,\big|\, \|x\| = 1\big\}(= B[0,1]\backslash B(0,1)$ which is closed and bounded), $g(a,\cdot)$ attains its minimum at some point $w \in \mathbb{S}^{n-1}$; that is,

$$g(a,u) \geqslant g(a,w) \qquad \forall\, u \in \mathbb{S}^{n-1}\,.$$

Let $\lambda = g(a,w) = (D^k f)(a)(w,\cdots,w) > 0$. Since f is of class $\mathscr{C}^k$, there exists $0 < \delta < r$ such that

$$\|(D^k f)(x) - (D^k f)(a)\| < \frac{\lambda}{2} \quad \text{whenever} \quad x \in B(a,\delta)\,.$$

Let $x \in B(a,\delta)\backslash\{a\}$ be given. By Taylor's Theorem there exists $c \in \overline{xa}$ (so that $c \in B(a,\delta)$) such that

$$f(x) = f(a) + \sum_{\ell=1}^{k-1} \frac{1}{\ell!}(D^\ell f)(a)(\overbrace{x-a,\cdots,x-a}^{\ell \text{ copies of } x-a}) + \frac{1}{k!}(D^k f)(c)(\overbrace{x-a,\cdots,x-a}^{k \text{ copies of } x-a})\,.$$

Since $(D^\ell f)(a)(u,u,\cdots,u) = 0$ for $1 \leqslant j \leqslant k-1$, we conclude that

$$f(x) = f(a) + \frac{1}{k!}(D^k f)(c)(x-a,x-a,\cdots,x-a) = f(a) + \frac{1}{k!}g(c,x-a)\,.$$

Note that (0.1) implies that

$$\Big|g\big(c,\frac{x-a}{\|x-a\|}\big) - g\big(a,\frac{x-a}{\|x-a\|}\big)\Big| \leqslant \big\|(D^k f)(c) - (D^k f)(a)\big\| < \frac{\lambda}{2}\,;$$

thus

$$g\big(c,\frac{x-a}{\|x-a\|}\big) > g\big(a,\frac{x-a}{\|x-a\|}\big) - \frac{\lambda}{2} \geqslant \frac{\lambda}{2}\,.$$

By the fact that $g(c,x-a) = g\big(c,\frac{x-a}{\|x-a\|}\big)\|x-a\|^k$, we conclude that

$$f(x) > f(a) + \frac{\lambda}{2k!}\|x-a\|^k \qquad \forall\, x \in B(a,\delta)\backslash\{a\}\,;$$

thus $f(x) \geqslant f(a)$ for all $x \in B(a,\delta)$. □