Exercise Problem Sets 14

Problem 1. Let $(X, \|\cdot\|_X)$ be a normed space, $A \subseteq X$ and $f : A \to \mathbb{R}$ be a function. f is said to attain a local extremum at $a \in A$ if there exists $r > 0$ such that either $f(x) \leqslant f(a)$ for all $x \in B(a, r) \cap A$ or $f(x) \geqslant f(a)$ for all $x \in B(a, r) \cap A$. Show that if f attains a local extremum at an interior point $a \in \mathring{A}$ and f is differentiable at a, then $(Df)(a)$ is the zero map in $\mathscr{B}(X, \mathbb{R})$.

Proof. Suppose that f attains an extremum at $a \in \mathring{A}$. There exists $r > 0$ such that $f(x) \leqslant f(a)$ for all $x \in B(a, r)$ or $f(x) \geqslant f(a)$ for all $x \in B(a, r)$. Since f is differentiable at a, by Remark 5.15 in the lecture note for all $\|v\|_X = 1$,

$$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = (Df)(a)(v).$$

Since f attains a local extremum at a, the function $g : (-r, r) \to \mathbb{R}$ defined by $g(t) = f(a + tv)$ attains a local maximum at $t = 0$; thus

$$0 = g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = (Df)(a)(v).$$

Since the identity above holds for all unit vector v, we find that $(Df)(a)$ is the zero map. $\quad\square$

Problem 2. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$ are bounded on U; that is, there exists a real number $M > 0$ such that

$$\left| \frac{\partial f}{\partial x_j}(x) \right| \leqslant M \qquad \forall\, x \in U \text{ and } j = 1, \cdots, n.$$

Show that f is continuous on U.
Hint: Mimic the proof of Theorem 5.40 in the lecture note.

Proof. Assume that $\left| \dfrac{\partial f}{\partial x_i}(x) \right| \leqslant M$ for all $x \in U$ and $1 \leqslant i \leqslant n$. Let $a \in U$ be given. Then there exists $r > 0$ such that $B(a, r) \subseteq U$. For $x \in B(a, r)$, let $k = x - a$. Then

$$
\begin{aligned}
\left| f(x) - f(a) \right| &= \left| f(a_1 + k_1, a_2 + k_2, \cdots, a_n + k_n) - f(a_1, a_2, \cdots, a_n) \right| \\
&= \left| \sum_{j=1}^{n} \left[f(a_1, \cdots, a_{j-1}, a_j + k_j, \cdots, a_n + k_n) - f(a_1, \cdots, a_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) \right] \right| \\
&\leqslant \sum_{j=1}^{n} \left| f(a_1, \cdots, a_{j-1}, a_j + k_j, \cdots, a_n + k_n) - f(a_1, \cdots, a_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) \right|.
\end{aligned}
$$

By the Mean Value Theorem, for each $1 \leqslant j \leqslant n$ there exists $\theta_j \in (0, 1)$ such that

$$
\begin{aligned}
&\left| f(a_1, \cdots, a_{j-1}, a_j + k_j, \cdots, a_n + k_n) - f(a_1, \cdots, a_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) \right. \\
&\qquad = \frac{\partial f}{\partial x_j}(a_1, \cdots, a_{j-1}, a_j + \theta_j k_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) k_j\,;
\end{aligned}
$$

thus

$$\left| f(a_1, \cdots, a_{j-1}, a_j + k_j, \cdots, a_n + k_n) - f(a_1, \cdots, a_j, a_{j+1} + k_{j+1}, \cdots, a_n + k_n) \right| \leqslant M |k_j| \,.$$

Therefore, if $x \in B(a, r)$,

$$\left| f(x) - f(a) \right| = \sum_{j=1}^n M |k_j| \leqslant M \sqrt{n} \Big(\sum_{j=1}^n |k_j|^2 \Big)^{\frac{1}{2}} = \sqrt{n} M \|x - a\|_{\mathbb{R}^n} \,.$$

This shows that f is continuous at a. □

Problem 3. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$. Show that f is differentiable at $a \in U$ if and only if there exists a vector-valued function $\varepsilon : U \to \mathbb{R}^n$ such that

$$f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) = \varepsilon(x) \cdot (x - a)$$

and $\varepsilon(x) \to 0$ as $x \to a$.

Proof. "⇒" Suppose that f is differentiable at a. Define $\varepsilon : U \to \mathbb{R}^n$ by

$$\varepsilon(x) = \begin{cases} \Big[f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \Big] \dfrac{x - a}{\|x - a\|^2} & \text{if } x \neq a \,, \\[2ex] 0 & \text{if } x = a \,. \end{cases}$$

Then for $x \neq a$,

$$\left| \varepsilon(x) \right| \leqslant \frac{\left| f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \right|}{\|x - a\|}$$

which, by the differentiability of f at a, implies that

$$\lim_{x \to a} \left| \varepsilon(x) \right| = 0 \,.$$

Moreover,

$$\varepsilon(x) \cdot (x - a) = f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \,.$$

"⇐" Suppose that there exists a vector-valued function $\varepsilon : U \to \mathbb{R}^n$ such that

$$f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) = \varepsilon(x) \cdot (x - a)$$

and $\varepsilon(x) \to 0$ as $x \to a$. Then for $x \neq a$, the Cauchy-Schwarz inequality implies that

$$\frac{\left| f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \right|}{\|x - a\|} = \frac{|\varepsilon(x) \cdot (x - a)|}{\|x - a\|} \leqslant \|\varepsilon(x)\| \,;$$

thus

$$\lim_{x \to a} \frac{\left| f(x) - f(a) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \right|}{\|x - a\|} = 0 \,.$$

Therefore, f is differentiable at a with $[(Df)(a)] = \Big[\dfrac{\partial f}{\partial x_1}(a), \cdots, \dfrac{\partial f}{\partial x_n}(a) \Big]$. □

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

and $u \in \mathbb{R}^2$ be a unit vector. Show that the directional derivative of f at the origin exists in all direction, and

$$(D_u f)(0, 0) = \left(\frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0) \right) \cdot u.$$

Is f differentiable at $(0, 0)$?

Solution. Let $u = (\cos \theta, \sin \theta)$ be a unit vector. Then the directional derivative of f at $(0, 0)$ in direction u is

$$(D_u f)(0, 0) = \lim_{t \to 0^+} \frac{f(t \cos \theta, t \sin \theta) - f(0, 0)}{t} = \lim_{t \to 0^+} \frac{t^4 \cos^3 \theta \sin \theta}{t(t^4 \cos^4 \theta + t^2 \sin^2 \theta)}$$

$$= \lim_{t \to 0^+} \frac{t \cos^3 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta} = 0.$$

On the other hand,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0 \quad \text{and} \quad f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = 0;$$

thus we conclude that $(D_u f)(0, 0) = \big(f_x(0, 0), f_y(0, 0) \big) \cdot u$.

Since $f_x(0, 0) = f_y(0, 0) = 0$, if f is differentiable at $(0, 0)$, we must have

$$\lim_{(x,y) \to (0,0)} \frac{\left| f(x, y) - f(0, 0) - 0 \cdot (x - 0) - 0 \cdot (y - 0) \right|}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{|x^3 y|}{\sqrt{x^2 + y^2}(x^4 + y^2)} = 0;$$

however, by passing to the limit as $(x, y) \to (0, 0)$ along the curve $y = x^2$, we find that

$$0 = \lim_{x \to 0} \frac{|x^3 \cdot x^2|}{\sqrt{x^2 + x^4}(x^4 + x^4)} = \lim_{x \to 0} \frac{1}{2\sqrt{1 + x^2}} = \frac{1}{2},$$

a contradiction. Therefore, f is not differentiable at $(0, 0)$. □