Exercise Problem Sets 13

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Problem 1. Let $\{T_k\}_{k=1}^{\infty} \subseteq \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ be a sequence of bounded linear maps from $\mathbb{R}^n \to \mathbb{R}^m$. Prove that the following three statements are equivalent:

1. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\{T_k \boldsymbol{x}\}_{k=1}^{\infty}$ converges to $T\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$;

2. $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$;

3. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that for every compact $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists $N > 0$ such that

$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \varepsilon \quad \text{whenever} \quad \boldsymbol{x} \in K \;\text{ and }\; k \geqslant N\,.$$

Proof. "1 ⇒ 3" Let K be a compact set in $\mathbb{R}^n$, and $\varepsilon > 0$ be given. Then there exists $R > 0$ such that $K \subseteq B[0, R]$. By assumption, for each $1 \leqslant i \leqslant n$, there exist $N_i > 0$ such that

$$\|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \frac{\varepsilon}{Rn} \quad \text{whenever} \quad k \geqslant N_i\,.$$

For $\boldsymbol{x} \in \mathbb{R}^n$, write $\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n$. Then if $\boldsymbol{x} \in K$, $|x^{(i)}| \leqslant R$ for all $1 \leqslant i \leqslant n$. Therefore, if $\boldsymbol{x} \in K$ and $k \geqslant N \equiv \max\{N_1, \cdots, N_n\}$,

$$\begin{aligned}\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} &= \Big\|T_k\Big(\sum_{i=1}^n x^{(i)}\mathbf{e}_i\Big) - T\Big(\sum_{i=1}^n x^{(i)}\mathbf{e}_i\Big)\Big\|_{\mathbb{R}^m} = \Big\|\sum_{i=1}^n x^{(i)}\big(T_k\mathbf{e}_i - T\mathbf{e}_i\big)\Big\|_{\mathbb{R}^m} \\ &\leqslant \sum_{i=1}^n |x^{(i)}| \|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \sum_{i=1}^n R\frac{\varepsilon}{Rn} = \varepsilon\,.\end{aligned}$$

"3 ⇒ 2" Let $K = B[0, 1]$ (which is compact), and $\varepsilon > 0$ be given. By assumption there exists $N > 0$ such that

$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{\varepsilon}{3} \quad \text{whenever} \quad \boldsymbol{x} \in B[0,1] \;\text{ and }\; k \geqslant N\,.$$

If $k, \ell \geqslant N$ and $\boldsymbol{x} \in B[0, 1]$,

$$\|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} + \|T_\ell\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{2\varepsilon}{3}$$

which shows that

$$\|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = \sup_{\boldsymbol{x}\in B[0,1]} \|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \frac{2\varepsilon}{3} < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Therefore, $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$.

"2 ⇒ 1" This part is essentially identical to the proof of Proposition 5.8 in the lecture note (with $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$). □

Problem 2. Recall that $\mathcal{M}_{m\times n}$ is the collection of all $m\times n$ real matrices. For a given $A\in\mathcal{M}_{m\times n}$, define a function $f:\mathcal{M}_{n\times m}\to\mathbb{R}$ by

$$f(M)=\operatorname{tr}(AM)\,,$$

where tr is the trace operator which maps a square matrix to the sum of its diagonal entries. Show that $f\in\mathscr{B}(\mathcal{M}_{n\times m},\mathbb{R})$.

Hint: You may need the conclusion in Example 4.29 in the lecture note.

Proof. Let $A=[a_{ij}]_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}$ and $M=[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$. Then

$$\operatorname{tr}(AM)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}\,.$$

First we show that $f\in\mathscr{L}(\mathcal{M}_{n\times m},\mathbb{R})$. Let $M=[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$ and $N=[n_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}$ be matrices in $\mathcal{M}_{n\times m}$ and $c\in\mathbb{R}$. Then

$$\begin{aligned}f(cM+N)=\operatorname{tr}(A(cM+N))&=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}(cm_{ji}+n_{ji})=c\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}+\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}n_{ji}\\&=c\operatorname{tr}(AM)+\operatorname{tr}(AN)=cf(M)+f(N)\,.\end{aligned}$$

Let $\|\cdot\|:\mathcal{M}_{n\times m}\to\mathbb{R}$ be defined by

$$\big\|[m_{jk}]_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}\big\|=\sum_{j=1}^{n}\sum_{k=1}^{m}|m_{jk}|\,.$$

Then $\|\cdot\|$ is a norm on $\mathcal{M}_{n\times m}$, and

$$\sup_{\|M\|=1}|f(M)|=\sup_{\sum_{j=1}^{n}\sum_{k=1}^{m}|m_{jk}|=1}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}m_{ji}\Big|\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|<\infty\,;$$

thus $f:(\mathcal{M}_{n\times m},\|\cdot\|)\to(\mathbb{R},|\cdot|)$ is bounded. Let $|||\cdot|||$ be another norm on $\mathcal{M}_{n\times m}$. Since $\mathcal{M}_{n\times m}$ is finite dimensional vector spaces over $\mathbb{R}$, there exists c and C such that

$$c\|M\|\leqslant|||M|||\leqslant C\|M\|\qquad\forall\,M\in\mathcal{M}_{n\times m}\,.$$

Therefore, $\big\{M\in\mathcal{M}_{n\times m}\,\big|\,|||M|||\leqslant 1\big\}\subseteq\Big\{M\in\mathcal{M}_{n\times m}\,\Big|\,\|M\|\leqslant\frac{1}{c}\Big\}$

$$\sup_{|||M|||=1}|f(M)|\leqslant\sup_{\|M\|\leqslant 1/c}|f(M)|=\sup_{\|cM\|\leqslant 1}\frac{1}{c}|f(cM)|\leqslant\frac{1}{c}\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|<\infty\,;$$

thus $f:(\mathcal{M}_{n\times m},|||\cdot|||)\to\mathbb{R}$ is bounded. □

Problem 3. Let $\mathscr{P}([0,1))$ be the collection of all polynomials defined on $[0,1]$, and $\|\cdot\|_\infty$ be the max-norm defined by $\|p\|_\infty=\max\limits_{x\in[0,1]}|p(x)|$.

1. Show that the differential operator $\dfrac{d}{dx}:\mathscr{P}([0,1])\to\mathscr{P}([0,1])$ is linear.

2. Show that $\frac{d}{dx}:\big(\mathscr{P}([0,1]),\|\cdot\|_\infty\big)\to\big(\mathscr{P}([0,1]),\|\cdot\|_\infty\big)$ is unbounded; that is, show that

$$\sup_{\|p\|_\infty=1}\|p'\|_\infty=\infty\,.$$

Proof. 1. Let $p,q\in\mathscr{P}([0,1])$ and $c\in\mathbb{R}$. Then by the rule of differentiation,

$$\frac{d}{dx}(cp+q)(x)=cp'(x)+q'(x)=c\frac{d}{dx}p(x)+\frac{d}{dx}q(x)\,;$$

thus $\frac{d}{dx}:\mathscr{P}([0,1])\to\mathscr{P}([0,1])$ is linear.

2. Consider $p_n(x)=x^n$. Then $\|p_n\|_\infty=\max\limits_{x\in[0,1]}x^n=1$ for all $n\in\mathbb{N}$; however,

$$\|p_n'\|_\infty=\max_{x\in[0,1]}nx^{n-1}=n\qquad n\in\mathbb{N}\,;$$

thus $\sup\limits_{\|p\|_\infty=1}\|p'\|_\infty=\infty$. □

Problem 4. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be normed spaces, and $T\in\mathscr{B}(X,Y)$. Show that for all $\boldsymbol{x}\in X$ and $r>0$,

$$\sup_{\boldsymbol{x}'\in B(\boldsymbol{x},r)}\|T\boldsymbol{x}'\|_Y\geqslant r\|T\|_{\mathscr{B}(X,Y)}\,.$$

Hint: Prove and make use of the inequality $\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y,\|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\}\geqslant\|T\boldsymbol{\xi}\|_Y$ for all $\boldsymbol{\xi}\in Y$.

Proof. Let $\boldsymbol{x}\in X$ and $r>0$ be given. Then for all $\boldsymbol{\xi}\in B(0,r)$,

$$\begin{aligned}&\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y,\|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\}\\&\quad\geqslant\frac{1}{2}\Big[\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y+\|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\Big]\geqslant\frac{1}{2}\|T(\boldsymbol{x}+\boldsymbol{\xi})-T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y=\|T\boldsymbol{\xi}\|_Y\,.\end{aligned}$$

Therefore,

$$\sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)}\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y,\|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\}\geqslant\sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)}\|T\boldsymbol{\xi}\|_Y=r\|T\|_{\mathscr{B}(X,Y)}\,,$$

and the desired inequality follows from the fact that

$$\sup_{\boldsymbol{x}'\in B(\boldsymbol{x},r)}\|T\boldsymbol{x}'\|_Y=\sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)}\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y,\|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\}\,.$$

□

Problem 5. Let $(X,\|\cdot\|_X)$ be a Banach space, $(Y,\|\cdot\|_Y)$ be a normed space, and $\mathscr{F}\subseteq\mathscr{B}(X,Y)$ be a family of bounded linear maps from X to Y. Show that if $\sup\limits_{T\in\mathscr{F}}\|T\boldsymbol{x}\|_Y<\infty$ for all $x\in X$, then

$$\sup_{T\in\mathscr{F}}\|T\|_{\mathscr{B}(X,Y)}<\infty\,.$$

Hint: Suppose the contrary that there exists $\{T_n\}_{n=1}^\infty\subseteq\mathscr{F}$ such that $\|T_n\|_{\mathscr{B}(X,Y)}\geqslant4^n$. Using Problem **??** to choose a sequence $\{\boldsymbol{x}_n\}_{n=0}^\infty$, where $\boldsymbol{x}_0=\boldsymbol{0}$, such that

$$\boldsymbol{x}_n\in B(\boldsymbol{x}_{n-1},3^{-n})\qquad\text{and}\qquad\|T_n\boldsymbol{x}_n\|_Y\geqslant\frac{2}{3}\cdot3^{-n}\|T_n\|_{\mathscr{B}(X,Y)}\,.$$

Show that $\{\boldsymbol{x}_n\}_{n=1}^\infty$ converges to some point $\boldsymbol{x}\in X$ but $\{T_n\boldsymbol{x}\}_{n=1}^\infty$ is not bounded in Y.

Remark: The conclusion above is called the Uniform Boundedness Principle (or the Banach-Steinhaus Theorem). This is one of the fundamental results in functional analysis.

Proof. Suppose the contrary that $\sup\limits_{T\in\mathscr{F}}\|T\|_{\mathscr{B}(X,Y)}=\infty$. Then there exists $\{T_n\}_{n=1}^{\infty}\subseteq\mathscr{F}$ such that

$$\|T_n\|_{\mathscr{B}(X,Y)}\geqslant 4^n \qquad \forall\, n\in\mathbb{N}\,.$$

Let $\boldsymbol{x}_0=\mathbf{0}$. Define $r_n=3^{-n}$ and $\{\boldsymbol{x}_n\}_{n=1}^{\infty}\subseteq X$ so that

$$\boldsymbol{x}_n\in B(\boldsymbol{x}_{n-1},r_n) \qquad \text{and} \qquad \|T_n\boldsymbol{x}_n\|_Y\geqslant\frac{2}{3}r_n\|T_n\|_{\mathscr{B}(X,Y)}\,.$$

We note that such $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ exists because of Problem **??**. For $m>n$,

$$\begin{aligned}\|\boldsymbol{x}_n-\boldsymbol{x}_m\|_X &\leqslant \|\boldsymbol{x}_n-\boldsymbol{x}_{n+1}\|_X+\|\boldsymbol{x}_{n+1}-\boldsymbol{x}_{n+1}\|_X+\cdots+\|\boldsymbol{x}_{m-1}-\boldsymbol{x}_m\|_X\\ &\leqslant 3^{-(n+1)}+3^{-(n+2)}+\cdots+3^{-m}\leqslant 3^{-(n+1)}\Big(1+\frac{1}{3}+\cdots\Big)\leqslant\frac{1}{2}\cdot 3^{-n}\,;\end{aligned}$$

thus $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X,\|\cdot\|_X)$ is complete, $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ converges to some point $\boldsymbol{x}\in X$, and $\|\boldsymbol{x}-\boldsymbol{x}_n\|_X\leqslant\frac{1}{2}\cdot 3^{-n}$. Therefore,

$$\begin{aligned}\|T_n\boldsymbol{x}\|_Y &\geqslant \|T_n\boldsymbol{x}_n\|_Y-\|T_n(\boldsymbol{x}-\boldsymbol{x}_n)\|_Y\geqslant\frac{2}{3}r_n\|T_n\|_{\mathscr{B}(X,Y)}-\|T_n\|_{\mathscr{B}(X,Y)}\|\boldsymbol{x}-\boldsymbol{x}_n\|_X\\ &\geqslant\Big(\frac{2}{3}-\frac{1}{2}\Big)\|T_n\|_{\mathscr{B}(X,Y)}3^{-n}=\frac{1}{6}\|T_n\|_{\mathscr{B}(X,Y)}3^{-n}\geqslant\frac{1}{6}\cdot\Big(\frac{4}{3}\Big)^n\end{aligned}$$

so that $\sup\limits_{n\in\mathbb{N}}\|T_n\boldsymbol{x}\|_Y=\infty$, a contradiction. ◻

Problem 6. Let $X=\mathcal{M}_{n\times m}$, the collection of all $n\times m$ real matrices, equipped with the Frobenius norm $\|\cdot\|_F$ introduced in Problem 7 of Exercise 5, and $f:X\to\mathbb{R}$ be defined by $f(A)=\|A\|_F^2$. Show that f is differentiable on X and find $(Df)(A)$ for $A\in X$.

Proof. First we note that $f(A)=\operatorname{tr}(AA^{\mathrm{T}})$, where $\operatorname{tr}(M)$ denotes the trace of M if M is a square matrix. Let $A=[a_{ij}]\in X$. Then for $\delta A\in X$, we have

$$\begin{aligned}f(A+\delta A)-f(A)&=\operatorname{tr}\big[(A+\delta A)(A+\delta A)^{\mathrm{T}}\big]-\operatorname{tr}(AA^{\mathrm{T}})\\ &=\operatorname{tr}\big(AA^{\mathrm{T}}+A\delta A^{\mathrm{T}}+\delta AA^{\mathrm{T}}+\delta A\delta A^{\mathrm{T}}\big)-\operatorname{tr}(AA^{\mathrm{T}})\\ &=\operatorname{tr}(A\delta A^{\mathrm{T}})+\operatorname{tr}(\delta AA^{\mathrm{T}})+\operatorname{tr}(\delta A\delta A^{\mathrm{T}})\,.\end{aligned}$$

Define $L_A:X\to\mathbb{R}$ by $L(B)=\operatorname{tr}(AB^{\mathrm{T}})+\operatorname{tr}(BA^{\mathrm{T}})$. Then Problem **??** shows that $L\in\mathscr{B}(X,\mathbb{R})$. Therefore, by the fact that

$$\lim_{\delta A\to 0}\frac{\big|f(A+\delta A)-f(A)-L_A(\delta A)\big|}{\|\delta A\|_F}=\lim_{\delta A\to 0}\frac{\big|\operatorname{tr}(\delta A\delta A^{\mathrm{T}})\big|}{\|\delta A\|_F}=\lim_{\delta A\to 0}\frac{\|\delta A\|_F^2}{\|\delta A\|_F}=\lim_{\delta A\to 0}\|\delta A\|_F=0\,,$$

we conclude that f is differentiable at A and $(Df)(A)=L_A$. ◻

Problem 7. Let $\|\cdot\|_F$ denote the Frobenius norm of matrices given in Problem 7 of Exercise 5. For an $m\times n$ matrix $A=[a_{ij}]$, we look for an $m\times k$ matrix $C=[c_{ij}]$ and an $k\times n$ matrix $R=[r_{ij}]$, where $1\leqslant k\leqslant\min\{m,n\}$, such that $\|A-CR\|_F^2$ is minimized. This is to minimize the function

$$f(C,R)=\|A-CR\|_F^2=\operatorname{tr}((A-CR)(A-CR)^{\mathrm{T}})=\sum_{i=1}^{n}\sum_{j=1}^{m}\big(a_{ij}-\sum_{\ell=1}^{k}c_{i\ell}r_{\ell j}\big)^2\,.$$

Show that if $C\in\mathbb{R}^{m\times k}$ and $R\in\mathbb{R}^{k\times n}$ minimize f, then C,R satisfy

$$(A-CR)R^{\mathrm{T}}=0 \qquad \text{and} \qquad C^{\mathrm{T}}(A-CR)=0\,.$$

Problem 8. Let $X = \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ equipped with norm $\|\cdot\|$, and $f : \mathrm{GL}(n) \to \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ be defined by $f(L) = L^{-2} \equiv L^{-1} \circ L^{-1}$. Show that f is differentiable on $\mathrm{GL}(n)$ and find $(Df)(L)$ for $L \in \mathrm{GL}(n)$.

Proof. Let $L \in \mathrm{GL}(n)$. By the fact that

$$K^{-1} - L^{-1} = -K^{-1}(K-L)L^{-1} \text{ and } K^{-2} - L^{-2} = -K^{-2}(K-L)L^{-1} - K^{-1}(K-L)L^{-2}\,,$$

we have

$$\begin{aligned} K^{-2} - L^{-2} &= -\big[L^{-2} - K^{-2}(K-L)L^{-1} - K^{-1}(K-L)L^{-2}\big](K-L)L^{-1} \\ &\quad - \big[L^{-1} - K^{-1}(K-L)L^{-1}\big](K-L)L^{-2} \\ &= -L^{-2}(K-L)L^{-1} - L^{-1}(K-L)L^{-2} + K^{-2}(K-L)L^{-1}(K-L)L^{-1} \\ &\quad + K^{-1}(K-L)L^{-2}(K-L)L^{-1} + K^{-1}(K-L)L^{-1}(K-L)L^{-2}\,; \end{aligned}$$

thus

$$\begin{aligned} &\|K^{-2} - L^{-2} + L^{-2}(K-L)L^{-1} + L^{-1}(K-L)L^{-2}\| \\ &\quad \leqslant \Big[\|K^{-2}\|\|L^{-1}\|^2 + 2\|K^{-1}\|\|L^{-1}\|\|L^{-2}\|\Big]\|K-L\|^2\,. \end{aligned} \tag{$\star$}$$

This motivates us to define $(Df)(L) \in \mathscr{B}(X, X)$ by

$$(Df)(L)(H) = -L^{-2}HL^{-1} - L^{-1}HL^{-2} \qquad \forall\, H \in X\,, \tag{$\diamond$}$$

and $(\star)$ implies that

$$\lim_{K \to L} \frac{\|f(K) - f(L) - (Df)(L)(K-L)\|}{\|K-L\|} = 0\,.$$

Therefore, f is differentiable on $\mathrm{GL}(n)$, and $(Df)(L)$ is given by $(\diamond)$. □

Problem 9. Let $X = \mathscr{C}([-,1,1];\mathbb{R})$ and $\|\cdot\|_X$ be defined by $\|f\|_X = \max\limits_{x\in[-1,1]} \big|f(x)\big|$, and $(Y, \|\cdot\|_Y) = (\mathbb{R}, |\cdot|)$. Consider the map $\delta : X \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is differentiable on X. Find $(D\delta)(f)$ (for $f \in \mathscr{C}([-1,1];\mathbb{R})$).

Proof. Let $f \in X$ be given. For $h \in X$, we have

$$\delta(f+h) - \delta f = \big(f(0) + h(0)\big) - f(0) = h(0) = \delta h\,;$$

thus we expect that $(D\delta)(f)(h) = \delta h$. We first show that $\delta \in \mathscr{B}(X, \mathbb{R})$.

1. For linearity, for $h_1, h_2 \in X$ and $c \in \mathbb{R}$, we have

$$\delta(ch_1 + h_2) = \big(ch_1 + h_2\big)(0) = ch_1(0) + h_2(0) = c\delta h_1 + \delta h_2\,.$$

2. For boundedness, if $\|h\|_X = 1$, then $\max\limits_{x\in[-1,1]} \big|h(x)\big| = 1$ so that

$$|\delta h| = \big|h(0)\big| \leqslant \max_{x\in[-1,1]} \big|h(x)\big| = 1 < \infty\,.$$

Having established that $\delta \in \mathscr{B}(X, \mathbb{R})$, we note that

$$\lim_{h \to 0} \frac{\big|\delta(f+h) - \delta f - \delta h\big|}{\|h\|_X} = \lim_{h \to 0} \frac{0}{\|h\|_X} = 0\,;$$

thus δ is differentiable at f (for all $f \in X$), and $(D\delta)(f) = \delta$ for all $f \in X$. □

Problem 10. Let $X = \mathscr{C}([a,b];\mathbb{R})$ and $\|\cdot\|_2$ be the norm induced by the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$. Define $I : X \to X$ by

$$I(f)(x) = \int_a^x f(t)^2\,dt \qquad \forall\, x \in [a,b]\,.$$

Show that I is differentiable on X, and find $(DI)(f)$.

Proof. Let $f \in X$ be given. For $h \in X$,

$$I(f+h)(x) - I(f)(x) = \int_a^x \big(f(t)+h(t)\big)^2\,dt - \int_a^x f(t)^2\,dt = \int_a^x \big[2f(t)h(t) + h(t)^2\big]\,dt\,; \qquad (\star\star)$$

thus we expect that

$$(DI)(f)(h)(x) = 2\int_a^x f(t)h(t)\,dt\,. \qquad (\diamond\diamond)$$

Define L by $(Lh)(x) = 2\int_a^x f(t)h(t)\,dt$.
Claim: $L \in \mathscr{B}(X,X)$.

1. For linearity, let $h_1, h_2 \in X$ and $c \in \mathbb{R}$. Then

$$L(ch_1 + h_2)(x) = 2\int_a^x f(t)\big(ch_1(t) + h_2(t)\big)\,dt = 2c\int_a^x f(t)h_1(t)\,dt + 2\int_a^x f(t)h_2(t)\,dt$$

   which shows that $L(ch_1 + h_2) = cL(h_1) + L(h_2)$.

2. Note that by the Cauchy-Schwarz inequality,

$$\Big|\int_a^x f(t)h(t)\,dt\Big| \leqslant \int_a^b |f(t)||h(t)|\,dt \leqslant \|f\|_2\|h\|_2\,;$$

   thus for $\|h\|_2 = 1$,

$$\|L(h)\|_2 = \Big[\int_a^b \Big(\int_a^x f(t)h(t)\,dt\Big)^2 dx\Big]^{\frac{1}{2}} \leqslant \Big(\int_a^b \|f\|_2^2\|h\|_2^2\,dx\Big)^{\frac{1}{2}} \leqslant \sqrt{b-a}\|f\|_2\,.$$

   Therefore,

$$\|L\| = \sup_{\|h\|_2=1} \|L(h)\|_2 \leqslant \sqrt{b-a}\|f\|_2 < \infty$$

   which shows that L is bounded.

Finally, using $(\star\star)$ we obtain that

$$\begin{aligned}\big\|I(f+h)-I(f)-L(h)\big\|_2 &= \Big[\int_a^b \Big(\int_a^x h(t)^2\,dt\Big)^2 dx\Big]^{\frac{1}{2}} \leqslant \Big[\int_a^b \Big(\int_a^b h(t)^2\,dt\Big)^2 dx\Big]^{\frac{1}{2}}\\ &= \Big[\int_a^b \|h\|_2^4\,dx\Big]^{\frac{1}{2}} = \sqrt{b-a}\|h\|_2^2\,;\end{aligned}$$

thus

$$\lim_{h\to 0}\frac{\big\|I(f+h)-I(f)-(DI)(f)(h)\big\|_2}{\|h\|_2} = 0\,.$$

Therefore, I is differentiable at f for all $f\in X$ and $(DI)(f)$ is given by $(\diamond\diamond)$. □

Problem 11. Let $X = \mathscr{D}([a,b];\mathbb{R})$, the collection of all piecewise continuously differentiable real-valued function, and $\|\cdot\|_2$ be the norm (on X) induced by the inner product

$$\langle f,g\rangle = \int_a^b \big[f(x)g(x)+f'(x)g'(x)\big]\,dx\,.$$

For $g\in\mathscr{C}([a,b];\mathbb{R})$, define $I: X\to\mathbb{R}$ by

$$I(f) = \int_a^b \big[g(t)-f'(t)\big]^2\,dt\,.$$

Show that I is differentiable on X, and find $(DI)(f)$.

Proof. Let $f,h\in X$, and L be defined by

$$L(h) = -2\int_a^b \big[g(t)-f'(t)\big]h'(t)\,dt\,.$$

Then $L: X\to\mathbb{R}$ is linear and the Cauchy-Schwartz inequality

$$\begin{aligned}|L(h)| &\leqslant 2\Big(\int_a^b \big|g(t)-f'(t)\big|^2\,dt\Big)^{\frac{1}{2}}\Big(\int_a^b \big|h'(t)\big|^2\,dt\Big)^{\frac{1}{2}}\\ &\leqslant 2\Big(\int_a^b \big|g(t)-f'(t)\big|^2\,dt\Big)^{\frac{1}{2}}\|h\|_X\end{aligned}$$

which implies that $L: X\to\mathbb{R}$ is bounded with

$$\|L\|_{\mathscr{B}(X,\mathbb{R})} \leqslant 2\Big(\int_a^b \big|g(t)-f'(t)\big|^2\,dt\Big)^{\frac{1}{2}} < \infty\,.$$

Therefore, by the fact that

$$\begin{aligned}&\big|I(f+h)-I(f)-L(h)\big|\\ &\quad= \Big|\int_a^b \Big[\big|g(t)-f'(t)-h'(t)\big|^2-\big|g(t)-f'(t)\big|^2\Big]\,dt + 2\int_a^b \big[g(t)-f'(t)\big]h'(t)\,dt\Big|\\ &\quad= \int_a^b \big|h'(t)\big|^2\,dt \leqslant \|h\|_X^2\,,\end{aligned}$$

we find that

$$\lim_{h\to 0}\frac{\big|I(f+h)-I(f)-L(h)\big|}{\|h\|_X} = 0\,.$$

Therefore, I is differentiable at f and $(DI)(f) = L$. □