Exercise Problem Sets 9

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Problem 1. A metric space (M, d) is said to be ***separable*** if there is a countable subset A which is dense in M. Show that every sequentially compact set is separable.
Hint: Consider the total boundedness using balls with radius $\dfrac{1}{n}$ for $n \in \mathbb{N}$.

Proof. Let K be a sequentially compact set in M. Then K is totally bounded; thus for each $n \in \mathbb{N}$ there exists a finite collection of points $\{x_1^{(n)}, x_2^{(n)}, \cdots, x_{N_n}^{(n)}\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^{N_n} B\big(x_j^{(n)}, \frac{1}{n}\big)\,.$$

Let $A = \bigcup\limits_{n=1}^{\infty} \{x_1^{(n)}, x_2^{(n)}, \cdots, x_{N_n}^{(n)}\}$. Then $A \subseteq K$ and A is countable since it is union of countably many finite sets. Moreover, for each $x \in K$ and $n \in \mathbb{N}$, there exists $1 \leqslant j \leqslant N_n$ such that $x \in B\big(x_j^{(n)}, \frac{1}{n}\big)$; thus for all $\varepsilon > 0$, $B(x, \varepsilon) \cap A \neq \varnothing$. Therefore, $x \in \bar{A}$, and this shows that $A \subseteq K \subseteq \bar{A}$; thus A is dense in K. □

Problem 2. Let (M, d) be a metric space.

1. Show that if M is complete and A is a totally bounded subset of M, then $\text{cl}(A)$ is sequentially compact.

2. Show that M is complete if and only if every totally bounded sequence has a convergent subsequence.

Proof. 1. Let $r > 0$ be given. Since A is totally bounded, there exist $x_1, x_2, \cdots, x_N \in M$ such that

$$A \subseteq \bigcup_{j=1}^{N} B\big(x_j, \frac{r}{2}\big)\,. \qquad (\star)$$

Note that for all $x \in M$, $B\big(x, \frac{r}{2}\big) \subseteq B\big[x, \frac{r}{2}\big]$ which further implies that

$$\text{cl}\big(B\big(x, \frac{r}{2}\big)\big) \subseteq B\big[x, \frac{r}{2}\big] \subseteq B(x, r) \qquad \forall\, x \in M\,.$$

Therefore, $(\star)$ and Problem 2 in Exercise 8 imply that

$$\bar{A} \subseteq \text{cl}\Big(\bigcup_{j=1}^{N} B\big(x_j, \frac{r}{2}\big)\Big) = \bigcup_{j=1}^{N} \text{cl}\big(B\big(x_j, \frac{r}{2}\big)\big) \subseteq \bigcup_{j=1}^{N} B(x_j, r)\,.$$

This shows that $\bar{A}$ is totally bounded. By the fact that (M, d) is complete, $\bar{A}$ is complete; thus $\bar{A}$ is sequentially compact.

2. "$\Rightarrow$" Let $\{x_n\}_{n=1}^\infty$ be a totally bounded subsequence. Define $A = \{x_n \,|\, n \in \mathbb{N}\}$. Then A is totally bounded, and (part of the proof of 1 shows that $\bar{A}$ is totally bounded); thus by the fact that M is complete 1 implies that $\bar{A}$ is sequentially compact. Since $\{x_n\}_{n=1}^\infty$ is a sequence in $\bar{A}$, we find that there exists a convergent subsequence of $\{x_n\}_{n=1}^\infty$ (that converges to a limit in $\bar{A}$).

   "$\Leftarrow$" By Proposition 2.58 in the lecture note it suffices to show that every Cauchy sequence is totally bounded.

   Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence, and $r > 0$ be given. Then there exists $N > 0$ such that $d(x_n, x_m) < r$ whenever $n, m \geqslant N$. In particular, $d(x_n, x_N) < r$ for all $n \geqslant N$ which further implies that $\{x_n\}_{n=N}^\infty \subseteq B(x_{N}, r)$. Therefore, $\{x_n\}_{n=1}^\infty \subseteq \bigcup_{n=1}^{N} B(x_n, r)$; thus $\{x_n\}_{n=1}^\infty$ is totally bounded. □

Alternative proof of 2 *of Problem* 2.
"$\Rightarrow$" Let $\{x_n\}_{n=1}^\infty$ be a totally bounded subsequence. Define $A = \{x_n \,|\, n \in \mathbb{N}\}$. Then A is totally bounded; thus by the fact that M is complete 1 implies that $\bar{A}$ is sequentially compact. Since $\{x_n\}_{n=1}^\infty$ is a sequence in $\bar{A}$, we find that there exists a convergent subsequence of $\{x_n\}_{n=1}^\infty$ (that converges to a limit in $\bar{A}$).
"$\Leftarrow$" By Proposition **??** it suffices to show that every Cauchy sequence is totally bounded.

Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence. If $\{x_n\}_{n=1}^\infty$ is not totally bounded, there exists $r > 0$ such that no finite collection of open balls with radius r can be a cover of $\{x_n\}_{n=1}^\infty$. Let $n_1 = 1$, and n_2 be the least integer satisfying $x_{n_2} \notin B(x_{n_1}, r)$, and n_3 be the least integer which is outside $B(x_{n_1}, r) \cup B(x_{n_2}, r)$. We continue this process and obtain $n_1 < n_2 < n_3 < \cdots$ such that

(a) $n_1 = 1$; (b) $x_{n_{k+1}} \notin \bigcup_{j=1}^{k} B(x_{n_j}, r)$ for all $k \in \mathbb{N}$.

However, this implies that there exists no $N > 0$ such that $d(x_n, x_m) < r$ for all $n, m \geqslant N$, a contradiction to that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. □

Problem 3. Let $\{x_k\}_{k=1}^\infty$ be a convergent sequence in a metric space, and $x_k \to x$ as $k \to \infty$. Show that the set $A \equiv \{x_1, x_2, \cdots, \} \cup \{x\}$ is sequentially compact.

Proof. See Example 3.57 in the lecture note. □

Problem 4. 1. Show the so-called "***Finite Intersection Property***":

> Let (M, d) be a metric space, and K be a subset of M. Then K is compact if and if for any family of closed subsets $\{F_\alpha\}_{\alpha \in I}$, we have
> $$K \cap \bigcap_{\alpha \in I} F_\alpha \neq \varnothing$$
> whenever $K \cap \bigcap_{\alpha \in J} F_\alpha \neq \varnothing$ for all $J \subseteq I$ satisfying $\#J < \infty$.

2. Show the so-called "***Nested Set Properpty***":

> Let (M, d) be a metric space. If $\{K_n\}_{n=1}^{\infty}$ is a sequence of non-empty compact sets in M such that $K_j \supseteq K_{j+1}$ for all $j \in \mathbb{N}$, then there exists at least one point in $\bigcap\limits_{j=1}^{\infty} K_j$; that is,
> $$\bigcap_{j=1}^{\infty} K_j \neq \varnothing .$$

Proof. 1. Suppose the contrary that $K \cap \bigcap_{\alpha \in I} F_\alpha = \varnothing$ for some family of closed subsets $\{F_\alpha\}_{\alpha \in I}$ satisfying that

$$K \cap \bigcap_{\alpha \in J} F_\alpha \neq \varnothing \text{ for all } J \subseteq I \text{ satisfying } \#J < \infty.$$

Then

$$K \subseteq \Big(\bigcap_{\alpha \in I} F_\alpha\Big)^{\complement} = \bigcup_{\alpha \in I} F_\alpha^{\complement}.$$

For each $\alpha \in I$, F_α is closed; thus the statement above shows that $\{F_\alpha^{\complement}\}_{\alpha \in I}$ is an open cover of K; thus the compactness of K provides a finite collection $F_{\alpha_1}, \cdots, F_{\alpha_N}$, where $\alpha_j \in I$ for all $1 \leqslant j \leqslant N$, such that

$$K \subseteq \bigcup_{j=1}^{N} F_{\alpha_j}^{\complement} = \Big(\bigcap_{j=1}^{N} F_{\alpha_j}\Big)^{\complement}.$$

which implies that $K \cap \bigcap\limits_{j=1}^{N} F_{\alpha_j} = \varnothing$, a contradiction.

2. Let $K = K_1$, and $F_j = K_j$ for all $j \in \mathbb{N}$. Then for any finite subset J of $\mathbb{N}$,

$$K \cap \bigcap_{j \in J} F_j = K_{\max J} \neq \varnothing ;$$

thus 1 implies that $K \cap \bigcap\limits_{j \in \mathbb{N}} F_j \neq \varnothing$. □

Problem 5. Let (M, d) be a metric space, and M itself is a sequentially compact set. Show that if $\{F_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that $\text{int}(F_k) = \varnothing$, then $M \backslash \bigcup\limits_{k=1}^{\infty} F_k \neq \varnothing$.

Proof. Let $U_k = F_k^{\complement}$. Since $\mathring{F}_k = \varnothing$ and F_k is closed, $\partial F_k = \overline{F_k} \backslash \mathring{F}_k = F_k$. Therefore, if $x \in F_k$ then $x \in \overline{U_k}$ while if $x \notin F_k$, then $x \in U_k$. In other words, every point $x \in M$ belongs to $\overline{U_k}$ so that we have $U_k \subseteq M \subseteq \overline{U_k}$ for all $k \in \mathbb{N}$; that is, U_k is dense in M for all $k \in \mathbb{N}$.

Claim: $\bigcap\limits_{k=1}^{\infty} U_k$ is dense in M.

Proof of claim: It suffices to show that $B(x, r) \cap \bigcap\limits_{k=1}^{\infty} U_k \neq \varnothing$ for all $x \in M$ and $r > 0$ $\big($for this shows that every $x \in M$ is in the closure of $\bigcap\limits_{k=1}^{\infty} U_k\big)$.

Let $x \in M$ and $r > 0$ be given. Since U_1 is dense in M, $B(x, r) \cap U_1 \neq \varnothing$. Let $x_1 \in B(x, r) \cap U_1$. Since $B(x, r) \cap U_1$ is open, there exists $r_1 > 0$ such that $B(x_1, 2r_1) \subseteq B(x, r) \cap U_1$. Since U_2 is dense in M, $B(x_1, r_1) \cap U_2 \neq \varnothing$. Let $x_2 \in B(x_1, r_1) \cap U_2$. By the fact that $B(x_1, r_1) \cap U_2$ is open, there exists $r_2 > 0$ such that $B(x_2, 2r_2) \subseteq B(x_1, r_1) \cap U_2$. Continuing this process, we obtain sequences $\{x_k\}_{k=1}^{\infty}$ in M and $\{r_k\}_{k=1}^{\infty}$ of positive numbers such that

$$B(x_k, 2r_k) \subseteq B(x_{k-1}, r_{k-1}) \cap U_k \qquad \forall\, k \in \mathbb{N}, \text{ where } x_0 = x \text{ and } r_0 = r\,.$$

Since $B[x_k, r_k]$ is a closed subset of a (sequentially) compact set M, $B[x_k, r_k]$ is itself a (sequentially) compact set. Moreover,

$$B[x_k, r_k] \subseteq B(x_k, 2r_k) \subseteq B(x_{k-1}, r_{k-1}) \cap U_k \subseteq B[x_{k-1}, r_{k-1}]\,,$$

so $\{B[x_k, r_k]\}_{k=1}^{\infty}$ is a nested sequence of compact sets. By the nested set property (2 of Problem 4), $\bigcap\limits_{k=1}^{\infty} B[x_k, r_k] \neq \varnothing$. Therefore, by the fact that

$$\begin{aligned} B(x,r) \cap \bigcap_{k=1}^{\infty} U_k &= B(x,r) \cap U_1 \cap \bigcap_{k=2}^{\infty} U_k \supseteq B(x_1, 2r_1) \cap \bigcap_{k=2}^{\infty} U_k \supseteq B[x_1, r_1] \cap \bigcap_{k=2}^{\infty} U_k \\ &\supseteq B[x_1, r_1] \cap B(x_1, r_1) \cap \bigcap_{k=2}^{\infty} U_k \supseteq B[x_1, r_1] \cap B(x_1, r_1) \cap U_2 \cap \bigcap_{k=3}^{\infty} U_k \\ &\supseteq B[x_1, r_1] \cap B[x_2, r_2] \cap \bigcap_{k=3}^{\infty} U_k \supseteq \cdots \supseteq \bigcap_{k=1}^{\infty} B[x_k, r_k] \neq \varnothing\,. \end{aligned}$$

Therefore, every ball intersects $\bigcap\limits_{k=1}^{\infty} U_k$ which concludes the claim. □

Having established the claim, the desired conclusion follows from the fact that a dense subset of a non-empty metric space cannot be empty. □

Problem 6. Let $M = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, x^2 + y^2 \leqslant 1\big\}$ with the standard metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\,.$$

Show that $A \subseteq M$ is sequentially compact if and only if A is closed.

Problem 7. 1. Let $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap\limits_{k=1}^{\infty} \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^{\infty}$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and $\operatorname{diam}(K_j) \to 0$ as $j \to \infty$, where

$$\operatorname{diam}(K_j) = \sup\big\{d(x,y) \,\big|\, x, y \in K_j\big\}\,.$$

Show that there is exactly one point in $\bigcap\limits_{j=1}^{\infty} K_j$.

Proof. 1. By 2, it suffices to show that $\bar{A}_k$ is non-empty compact set for all $k \in \mathbb{N}$ and $\{\bar{A}_k\}_{k=1}^{\infty}$ is a nested set satisfying $\operatorname{diam}(\bar{A}_k) \to 0$ as $k \to \infty$. Note that in class we have shown that the set

$\{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \cdots\right\}$ is compact, and similar proof shows that $A_k \cup \{x\}$ is compact; thus $\bar{A}_k = A_k \cup \{x\}$. Therefore, $\{\bar{A}_k\}_{k=1}^{\infty}$ is a nested set.

Let $\varepsilon > 0$ be given. Since $\{x_k\}_{k=1}^{\infty}$ converges to x, there exists $N > 0$ such that $d(x_k, x) < \frac{\varepsilon}{3}$ whenever $k \geqslant N$. Then

$$d(y, z) < \frac{2\varepsilon}{3} \qquad \forall\, y, z \in A_N\,;$$

thus for $j \geqslant N$,

$$\operatorname{diam}(K_j) \leqslant \frac{2\varepsilon}{3} < \varepsilon$$

which implies that $\operatorname{diam}(K_j) \to 0$ as $j \to \infty$.

2. First, by the nested set property, $\bigcap\limits_{j=1}^{\infty} K_j \neq \varnothing$. Assume that $x, y \in \bigcap\limits_{j=1}^{\infty} K_j$. Then $x, y \in K_j$ for all $j \in \mathbb{N}$; thus

$$0 \leqslant d(x, y) \leqslant \operatorname{diam}(K_j) \qquad \forall\, j \in \mathbb{N}\,.$$

By the assumption that $\operatorname{diam}(K_j) \to 0$ as $j \to \infty$, we conclude that $d(x, y) = 0$; thus by the property of the metric, $x = y$. □

Problem 8. Let (M, d) be a metric space, and A be a subset of M satisfying that every sequence in A has a convergent subsequence (with limit in M). Show that A is pre-compact.
Remark: Together with Remark 3.61 in the lecture note, we conclude that a subset A is pre-compact if and only if A has the property that "every sequence in A has a convergent subsequence".

Proof. Let A be a subset of M satisfying that every sequence in A has a convergent subsequence, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\bar{A}$. Since $\bar{A}$ is the collection of limit points of A, each x_n is a limit point of A; thus for each $n \in \mathbb{N}$ there exists $y_n \in A$ such that $d(x_n, y_n) < \frac{1}{n}$. Using the property of A, there exists a convergent subsequence $\{y_{n_j}\}_{j=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ with limit y. By the fact that $\{y_n\}_{n=1}^{\infty} \subseteq A$, we must have $y \in \bar{A}$. Next we show that $\lim\limits_{j\to\infty} x_{n_j} = y$.

Let $\varepsilon > 0$ be given. Choose $K > 0$ so that $\frac{1}{K} < \frac{\varepsilon}{2}$. Moreover, since $\{y_{n_j}\}_{j=1}^{\infty}$ converges to y, there exists $J > 0$ such that

$$d(y_{n_j}, y) < \frac{\varepsilon}{2} \quad \text{whenever} \quad j \geqslant J\,.$$

Let $N = \max\{K, J\}$. Then if $j \geqslant N$, we must have

$$d(x_{n_j}, y_{n_j}) < \frac{1}{n_j} \leqslant \frac{1}{j} < \frac{\varepsilon}{2} \quad \text{and} \quad d(y_{n_j}, y) < \frac{\varepsilon}{2}$$

so that

$$d(x_{n_j}, y) \leqslant d(x_{n_j}, y_{n_j}) + d(y_{n_j}, y) < \varepsilon \quad \text{whenever} \quad j \geqslant N\,.$$ □

Problem 9. Let (M, d) be a metric space, and $A \subseteq M$. Show that A is disconnected (not connected) if and only if there exist non-empty closed set F_1 and F_2 such that

1. $A \cap F_1 \cap F_2 = \varnothing$; 2. $A \cap F_1 \neq \varnothing$; 3. $A \cap F_2 \neq \varnothing$; 4. $A \subseteq F_1 \cup F_2$.

Proof. By definition, A is disconnected if (and only if) there exist non-empty open set U_1 and U_2 such that

$$\text{(a) } A \cap U_1 \cap U_2 = \varnothing\,, \quad \text{(b) } A \cap U_1 \neq \varnothing\,, \quad \text{(c) } A \cap U_2 \neq \varnothing\,, \quad \text{(d) } A \subseteq U_1 \cup U_2\,.$$

Therefore, A is disconnected if and only if there exist non-empty closed set $F_1 \equiv U_1^{\complement}$ and $F_2 \equiv U_2^{\complement}$ such that

$$\text{(i) } A \cap F_1^{\complement} \cap F_2^{\complement} = \varnothing\,, \quad \text{(ii) } A \cap F_1^{\complement} \neq \varnothing\,, \quad \text{(iii) } A \cap F_2^{\complement} \neq \varnothing\,, \quad \text{(iv) } A \subseteq F_1^{\complement} \cup F_2^{\complement}\,.$$

Note that (i) above is equivalent to that $A \subseteq F_1 \cup F_2$, while (iv) above is equivalent to that $A \cap F_1 \cap F_2 = \varnothing$. Moreover, note that if A, B, C are sets satisfying $A \cap B \cap C = \varnothing$, $A \cap B \neq \varnothing$ and $A \cap C \neq \varnothing$, then

$$\varnothing \neq A \cap B \subseteq A \cap C^{\complement} \quad \text{and} \quad \varnothing \neq A \cap C \subseteq A \cap B^{\complement}\,.$$

Therefore, (a), (b) and (c) above imply 2 and 3 above, while (i) together with 2 and 3 above implies that (b) and (c); thus we establish that A is disconnected if and only if there exist non-empty closed sets F_1 and F_2 such that

$$1.\ A \cap F_1 \cap F_2 = \varnothing\,; \quad 2.\ A \cap F_1 \neq \varnothing\,; \quad 3.\ A \cap F_2 \neq \varnothing\,; \quad 4.\ A \subseteq F_1 \cup F_2\,.$$

□

Problem 10. Prove that if A is connected in a metric space (M, d) and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Proof. Suppose the contrary that B is disconnected. Then Problem 9 implies that there exist two closed set F_1 and F_2 such that

$$1.\ B \cap F_1 \cap F_2 = \varnothing\,; \quad 2.\ B \cap F_1 \neq \varnothing\,; \quad 3.\ B \cap F_2 \neq \varnothing\,; \quad 4.\ B \subseteq F_1 \cup F_2\,.$$

Define $A_1 = F_1 \cap A$ and $A_2 = F_2 \cap A$. Then $A = A_1 \cup A_2$. If $A_1 = \varnothing$, then $A_2 = A$ which, together with 3 of Problem 6 in Exercise 7, implies that

$$B \subseteq \bar{A} = \bar{A}_2 \subseteq \bar{A} \cap \bar{F}_2 = \bar{A} \cap F_2$$

which implies that $B = B \cap F_2$. The fact that $B \cap F_1 \cap F_2 = \varnothing$ then implies that $B \cap F_1 \subseteq (B \cap F_2)^{\complement} = B^{\complement}$; thus $B \cap F_1 = \varnothing$, a contradiction. Therefore, $A_1 \neq \varnothing$. Similarly, $A_2 \neq \varnothing$. However, 3 of Problem 6 in Exercise 7 implies that

$$A_1 \cap \bar{A}_2 = A_1 \cap \operatorname{cl}(F_2 \cap A) \subseteq A_1 \cap \bar{F}_2 \cap \bar{A} = A_1 \cap F_2 \subseteq B \cap F_1 \cap F_2 = \varnothing$$

and

$$A_2 \cap \bar{A}_1 = A_2 \cap \operatorname{cl}(F_1 \cap A) \subseteq A_2 \cap \bar{F}_1 \cap \bar{A} = A_2 \cap F_1 \subseteq B \cap F_2 \cap F_1 = \varnothing\,,$$

a contradiction to the assumption that A is connected. □

Problem 11. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Suppose that A is connected and contain more than one point. Show that $A \subseteq A'$.

Proof. Suppose the contrary that there exists $x \in A\backslash A'$. Since $A\backslash A'$ is the collection of isolated point of A, there exists $r > 0$ such that $B(x,r) \cap A = \{x\}$. Let $U = B(x,r)$ and $V = B\big[x, \frac{r}{2}\big]^{\complement}$. Then

1. $A \cap U \cap V = \varnothing$.

2. $A \cap U = \{x\} \neq \varnothing$.

3. $A \cap V \supseteq A\backslash\{x\} \neq \varnothing$ since A contains more than one point.

4. $A \cap M = U \cup V$.

Therefore, A is disconnected, a contradiction. □

Problem 12. Show that the Cantor set C defined in Problem 9 of Exercise 8 is totally disconnected; that is, if $x, y \in C$, and $x \neq y$, then $x \in U$ and $y \in V$ for some open sets U, V separate C.

Proof. It suffices to show that for every $x, y \in C$, $x < y$, there exists $z \in C^{\complement}$ and $x < z < y$. Note that there exists $N > 0$ such that $|x - y| < \frac{1}{3^n}$ for all $n \geqslant N$. If $C = \bigcap\limits_{n=1}^{\infty} E_n$, where E_n is given in Problem 9 of Exercise 8. Then the length of each interval in E_n has length $\frac{1}{3^n}$; thus if $n \geqslant N$, the interval $[x, y]$ is not contained in any interval of E_n. In other words, there must be $z \in (x, y)$ such that $z \in E_n^{\complement}$. Since $E_n^{\complement} \subseteq C^{\complement}$, we establish the existence of $x < z < y$ such that $z \in C^{\complement}$. □

Problem 13. Let F_k be a nest of connected compact sets (that is, $F_{k+1} \subseteq F_k$ and F_k is connected for all $k \in \mathbb{N}$). Show that $\bigcap\limits_{k=1}^{\infty} F_k$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that F_k is a nest of closed connected sets.

Proof. Let $K = \bigcap\limits_{k=1}^{\infty} F_k$. Then the nested set property implies that $K \neq \varnothing$. Suppose the contrary that there exist open sets U and V such that

1. $K \cap U \cap V = \varnothing$, 2. $K \cap U \neq \varnothing$, 3. $K \cap V \neq \varnothing$, 4. $K \subseteq U \cup V$.

Define $K_1 = K \cap U^{\complement}$ and $K_2 = K \cap V^{\complement}$. Then K_1, K_2 are non-empty closed sets (**Check!!!**) of K such that

$$K = K_1 \cup K_2 \quad \text{and} \quad K_1 \cap K_2 = \varnothing\,.$$

In other words, K is the disjoint union of two compact subsets K_1 and K_2. By (5) of Problem 7, there exists $x_1 \in K_1$ and $x_2 \in K_2$ such that $d(x_1, x_2) = d(K_1, K_2)$. Since $K_1 \cap K_2 = \varnothing$, $\varepsilon_0 \equiv d(x_1, x_2) > 0$; thus the definition of the distance of sets implies that

$$\varepsilon_0 \leqslant d(x, y) \qquad \forall\, x \in K_1, y \in K_2\,.$$

Define $O_1 = \bigcup\limits_{x \in K_1} B\big(x, \frac{\varepsilon_0}{3}\big)$ and $O_2 = \bigcup\limits_{y \in K_2} B\big(y, \frac{\varepsilon_0}{3}\big)$. Note that

$$K_1 \subseteq O_1\,, \quad K_2 \subseteq O_2 \quad \text{and} \quad O_1 \cap O_2 = \varnothing\,.$$

Claim: There exists $n \in \mathbb{N}$ such that $F_n \subseteq O_1 \cup O_2$.

Proof. Suppose the contrary that for each $n_0 \in \mathbb{N}$, $F_{n_0} \nsubseteq O_1 \cup O_2$. Then

$$F_n \cap O_1^{\complement} \cap O_2^{\complement} = F_n \cap (O_1 \cup O_2)^{\complement} \neq \varnothing \qquad \forall\, n \in \mathbb{N}\,.$$

Since O_1 and O_2 are open, $F_n \cap O_1^{\complement} \cap O_2^{\complement}$ is a nest of non-empty compact sets; thus the nested set property shows that

$$K \cap O_1^{\complement} \cap O_2^{\complement} = \bigcap_{n=1}^{\infty} (F_n \cap O_1^{\complement} \cap O_2^{\complement}) \neq \varnothing\,;$$

thus $K \nsubseteq O_1 \cup O_2$, a contradiction. □

Having established the claim, by the fact that $K_1 \subseteq F_{n_0} \cap O_1$ and $K_2 \subseteq F_{n_0} \cap O_2$, we find that

$$F_{n_0} \cap O_1 \neq \varnothing \qquad \text{and} \qquad F_{n_0} \cap O_2 \neq \varnothing\,.$$

Together with the fact that $F_{n_0} \cap O_1 \cap O_2 = \varnothing$ and $F_{n_0} \subseteq O_1 \cup O_2$, we conclude that F_{n_0} is disconnected, a contradiction.

The compactness of F_n is crucial to obtain the result or we will have counter-examples. For example, let $F_k = \mathbb{R}^2 \backslash (-k, k) \times (-1, 1)$. Then clearly F_k is closed but not bounded (hence F_k is not compact). Moreover, $F_k \supseteq F_{k+1}$ for all $k \in \mathbb{N}$; thus $\{F_k\}_{k=1}^{\infty}$ is a nest of sets. However, $\bigcap\limits_{k=1}^{\infty} F_k = \mathbb{R}^2 \backslash \mathbb{R} \times (-1, 1)$ which is disconnected and is the union of two disjoint closed set $\mathbb{R} \times [1, \infty)$ and $\mathbb{R} \times (-\infty, -1]$. Therefore, if $\{F_k\}_{k=1}^{\infty}$ is a nest of closed connected sets, it is possible that $\bigcap\limits_{k=1}^{\infty} F_k$ is disconnected. □

Problem 14. Let $\{A_k\}_{k=1}^{\infty}$ be a family of connected subsets of M, and suppose that A is a connected subset of M such that $A_k \cap A \neq \varnothing$ for all $k \in \mathbb{N}$. Show that the union $\big(\bigcup\limits_{k \in \mathbb{N}} A_k\big) \cup A$ is also connected.

Proof. By the induction argument, it suffices to show that if A and B are connected sets and $A \cap B \neq \varnothing$, then $A \cup B$ is connected. Suppose the contrary that there exist open sets U and V such that

1. $\big(A \cup B\big) \cap U \cap V = \varnothing\,,$
2. $\big(A \cup B\big) \cap U \neq \varnothing\,,$
3. $\big(A \cup B\big) \cap V \neq \varnothing\,,$
4. $\big(A \cup B\big) \subseteq U \cup V.$

Note that 1 and 4 implies that $A \cap U \cap V = \varnothing$ and $A \subseteq U \cup V$; thus by the connectedness of A, either $A \cap U = \varnothing$ or $A \cap V = \varnothing$. W.L.O.G., we assume that $A \cap U = \varnothing$ so that $A \subseteq V$. Then 1 implies that $B \cap U \cap V = \varnothing$, 2 implies that $B \cap U \neq \varnothing$, and 4 implies that $B \subseteq U \cup V$. Next we show that $B \cap V \neq \varnothing$ to reach a contradiction (to that B is connected). Suppose the contrary that $B \cap V = \varnothing$. Then 3 implies that $A \cap B \subseteq A = A \cap V \neq \varnothing$ so that $B \cap V \supseteq A \cap B \neq \varnothing$, a contradiction. □

Problem 15. Let $A, B \subseteq M$ and A is connected. Suppose that $A \cap B \neq \varnothing$ and $A \cap B^{\complement} \neq \varnothing$. Show that $A \cap \partial B \neq \varnothing$.

Proof. Suppose the contrary that $A \cap \partial B = \varnothing$. Let $U = \text{int}(B)$ and $V = \text{int}(B^{\complement})$. If $\mathring{B} = \varnothing$, then $\partial B = \bar{B} \supseteq B$; thus the assumption that $A \cap B \neq \varnothing$ implies that $A \cap \partial B \neq \varnothing$. Similarly, if $\text{int}(B^{\complement}) = \varnothing$, then $A \cap \partial B \neq \varnothing$.

Now suppose that U and V are non-empty open sets. If $x \notin U \cup V$, then $x \in \partial B$; thus $(U \cup V)^{\complement} \subseteq \partial B$ and the assumption that $A \cap \partial B = \varnothing$ further implies that $A \subseteq U \cup V$. Moreover, $U \cap V = \varnothing$; thus $A \cap U \cap V = \varnothing$. Now we prove that $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$ to reach a contradiction.

Suppose the contrary that $A \cap U = \varnothing$. Then $A \cap B \subseteq A \cap \bar{B} = A \cap (U \cup \partial B) = \varnothing$, a contradiction. Therefore, $A \cap U = \varnothing$. Similarly, if $A \cap V = \varnothing$, $A \cap B^{\complement} \subseteq A \cap \overline{B^{\complement}} = A \cap (V \cup \partial B^{\complement}) = A \cap (V \cup \partial B) = \varnothing$, a contradiction. □

Problem 16. Let (M, d) be a metric space and A be a non-empty subset of M. A maximal connected subset of A is called a ***connected component*** of A.

1. Let $a \in A$. Show that there is a unique connected components of A containing a.

2. Show that any two distinct connected components of A are disjoint. Therefore, A is the disjoint union of its connected components.

3. Show that every connected component of A is a closed subset of A.

4. If A is open, prove that every connected component of A is also open. Therefore, when $M = \mathbb{R}^n$, show that A has at most countable infinite connected components.

5. Find the connected components of the set of rational numbers or the set of irrational numbers in $\mathbb{R}$.

Proof. 1. Let $\mathscr{F}$ be the family $\mathscr{F} = \{C \subseteq A \,|\, x \in C \text{ and } C \text{ is connected}\}$. We note that $\mathscr{F}$ is not empty since $\{x\} \in \mathscr{F}$. Let $B = \bigcup\limits_{C \in \mathscr{F}} C$. It then suffices to show that B is connected (since if so, then it is the maximal connected subset of A containing x).

Claim: A subset $A \subseteq M$ is connected if and only if every continuous function defined on A whose range is a subset of $\{0, 1\}$ is constant.

Proof. "⇒" Assume that A is connected and $f : A \to \{0, 1\}$ is a continuous function, and $\delta = 1/2$. Suppose the contrary that $f^{-1}(\{0\}) \neq \varnothing$ and $f^{-1}(\{1\}) \neq \varnothing$. Then $A = f^{-1}((-\delta, \delta))$ and $B = f^{-1}((1-\delta, 1+\delta))$ are non-empty set. Moreover, the continuity of f implies that A and B are open relative to A; thus there exist open sets U and V such that

$$f^{-1}((-\delta, \delta)) = U \cap A \qquad \text{and} \qquad f^{-1}((1-\delta, 1+\delta)) = V \cap A\,.$$

Then

$$(1)\quad A \cap U \cap V = f^{-1}((-\delta, \delta)) \cap f^{-1}((1-\delta, 1+\delta)) = \varnothing\,,$$

(2) $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$,

(3) $A \subseteq U \cup V$ since the range of f is a subset of $\{0,1\}$;

thus A is disconnect, a contradiction.

"$\Leftarrow$" Suppose the contrary that A is disconnected so that there exist open sets U and V such that

(1) $A \cap U \cap V = \varnothing$, (2) $A \cap U \neq \varnothing$, (3) $A \cap V \neq \varnothing$, (4) $A \subseteq U \cup V$.

Let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \cap U\,, \\ 1 & \text{if } x \in A \cap V\,. \end{cases}$$

We first prove that f is continuous on A. Let $a \in A$. Then $a \in A \cap U$ or $a \in A \cap V$. Suppose that $a \in A \cap U$. In particular $a \in U$; thus the openness of U provides $r > 0$ such that $B(a,r) \subseteq U$. Note that if $x \in B(a,r) \cap A$, then $x \in A \subseteq U$; thus

$$\big|f(x) - f(a)\big| = 0 \qquad \forall\, x \in B(a,r) \cap A$$

which shows the continuity of f at a. Similar argument can be applied to show that f is continuous at $a \in A \cap V$. □

Now let $f : B \to \{0,1\}$ be a continuous function. Let $y \in B$. Then $y \in C$ for some $C \in \mathscr{F}$. Since C is a connected set, $f : C \to \{0,1\}$ is a constant; thus by the fact that $x \in C$, we must have $f(x) = f(y)$. Therefore, $f(y) = f(x)$ for all $y \in B$; thus $f : B \to \{0,1\}$ is a constant. The claim then shows that B is connected.

2. By Problem 14, the union of two overlapping connected sets is connected; thus distinct connected components of A are disjoint.

3. Let C be a connected component of A.

Claim: $\bar{C} \cap A$ is connected.

Proof. Suppose the contrary that there exist open sets U and V such that

(1) $\bar{C} \cap A \cap U \cap V = \varnothing$, (2) $\bar{C} \cap A \cap U \neq \varnothing$, (3) $\bar{C} \cap A \cap V \neq \varnothing$, (4) $\bar{C} \cap A \subseteq U \cup V$.

Note that (1) and (4) implies that $C \cap U \cap V = \varnothing$ and $C \subseteq U \cup V$ since $C \subseteq \bar{C} \cap A$. If $C \cap U = \varnothing$, then $C \subseteq U^{\complement}$; thus the closedness of $U^{\complement}$ implies that $\bar{C} \subseteq U^{\complement}$ which shows that $\bar{C} \cap A \cap U = \varnothing$, a contradiction. Therefore, $C \cap U \neq \varnothing$. Similarly, $C \cap V \neq \varnothing$, so we establish that C is disconnected, a contradiction. □

Having established that $\bar{C} \cap A$ is connected, we immediately conclude that $C = \bar{C} \cap A$ since $C \subseteq \bar{C} \cap A$ and C is the largest connected component of A containing points in C.

4. Suppose that A is open and C is a connected component of A. Let $x \in C$. Then $x \in A$; thus there exists $r > 0$ such that $B(x,r) \subseteq A$. Note that $B(x,r)$ is a connected set and $B(x,r) \cap C \supseteq \{x\} \neq \varnothing$. Therefore, Problem 14 implies that $B(x,r) \cup C$ is a connected subset of A containing x. Since C is the largest connected subset of A containing x, we must have $B(x,r) \cup C = C$; thus $B(x,r) \subseteq C$.

   If $M = \mathbb{R}^n$, then each connected component contains a point whose components are all rational. Since $\mathbb{Q}^n$ is countable, we find that an open set in $\mathbb{R}^n$ has countable connected components.

5. In $(\mathbb{R}, |\cdot|)$ every connected set is an interval or a set of a single point. Since $\mathbb{Q}$ and $\mathbb{Q}^\complement$ do not contain any intervals, the connected component of $\mathbb{Q}$ and $\mathbb{Q}^\complement$ are points. $\square$