Exercise Problem Sets 8

Nov. 04. 2022

Definition 0.1. Let (M, d) be a normed vector space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A\backslash\{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.
2. A point $x \in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A\backslash\{x\}$ that converges to x.
3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Problem 1. Let (M, d) be a metric space, and A be a subset of M. Show that $A \supseteq A'$ if and only if A is closed.

Proof. "$\Leftarrow$" Note that 2 of Problem 5 of Exercise 7 implies that $\bar{A} \supseteq A'$; thus if A is closed, $A = \bar{A} \supseteq A'$.

"$\Rightarrow$" In 2 of Problem 5 of Exercise 7, we establish that $\bar{A} = A \cup A'$. Therefore, if $A \supseteq A'$, we have $\bar{A} = A \cup A' = A$ which shows that A is closed. □

Problem 2. Show that the derived set of a set (in a metric space) is closed.

Proof. Let (M, d) be a metric space, and A be a subset of M. The goal is to show that A' is closed (and this is equivalent of showing that $(A')^{\complement}$ is open). Let $y \notin A'$. Then there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) \cap (A\backslash\{y\}) = (B(y, \varepsilon)\backslash\{y\}) \cap A = \varnothing\,.$$

Then $A \subseteq \big(B(y, \varepsilon)\backslash\{y\}\big)^{\complement}$. Since

$$\big(B(y, \varepsilon)\backslash\{y\}\big)^{\complement} = \big(B(y, \varepsilon) \cap \{y\}^{\complement}\big)^{\complement} = B(y, \varepsilon)^{\complement} \cup \{y\}\,,$$

$\big(B(y, \varepsilon)\backslash\{y\}\big)^{\complement}$ is closed. Therefore, Theorem 3.5 in the lecture note implies that

$$\bar{A} \subseteq \big(B(y, \varepsilon)\backslash\{y\}\big)^{\complement} \quad \text{or equivalently,} \quad \bar{A} \cap B(y, \varepsilon)\backslash\{y\} = \varnothing\,.$$

Since $\bar{A} = A \cup A'$, the equality above implies that

$$A' \cap B(y, \varepsilon)\backslash\{y\} = \varnothing\,;$$

thus the fact that $y \notin A'$ implies that $B(y, \varepsilon) \cap A' = \varnothing$. □

Problem 3. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Complete the following.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.

2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.

3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$?

4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap\limits_{m=1}^{\infty} A^{(m)} = \varnothing$?

5. Let $A = \left\{\frac{1}{m} + \frac{1}{k} \,\middle|\, m-1 > k(k-1), m, k \in \mathbb{N}\right\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Proof. 1. See Problem 2 for that A' is closed for all $A \subseteq M$. Moreover, Problem 1 shows that $A \supseteq A'$ if A is closed (in fact, A is closed if and only if $A \supseteq A'$). Therefore, knowing that $A^{(m)}$ is closed for all $m \in \mathbb{N}$, we obtain that $A^{(m)} \supseteq A^{(m+1)}$ for all $m \in \mathbb{N}$.

2. Note that $A \backslash A'$ consists of all isolated points of A. For $m \in \mathbb{N}$, define $B^{(m-1)} = A^{(m-1)} \backslash A^{(m)}$. Then $B^{(m-1)}$ consists of isolated points of $A^{(m-1)}$; thus $B^{(m-1)}$ is countable for all $m \in \mathbb{N}$. Since for any subset A of M, we have

$$A \subseteq (A \backslash A') \cup A'$$

and equality holds if A is closed, 1 implies that

$$\begin{aligned} A &\subseteq (A \backslash A^{(1)}) \cup A^{(1)} = B^{(0)} \cup A^{(1)} = B^{(0)} \cup \left[\left(A^{(1)} \backslash A^{(2)}\right) \cup A^{(2)}\right] = B^{(0)} \cup B^{(1)} \cup A^{(2)} \\ &= \cdots = B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(m-1)} \cup A^{(m)} . \end{aligned}$$

If $A^{(m)}$ is countable, we find that A is a subset of a finite union of countable sets; thus A is countable.

4. By 2, if $A^{(m)}$ is countable for some $m \in \mathbb{N}$, then A is countable; thus if A is uncountable, $A^{(m)}$ must be uncountable for all $m \in \mathbb{N}$.

5. For each $k \in \mathbb{N}$, let $B_k = \left\{\frac{1}{m} + \frac{1}{k} \,\middle|\, m-1 > k(k-1), m, k \in \mathbb{N}\right\}$. Then $A = \bigcup\limits_{k=1}^{\infty} B_k$. Moreover, for each $k \in \mathbb{N}$,

$$\sup B_k = \frac{1}{k(k-1)+2} + \frac{1}{k} \qquad \text{and} \qquad \inf B_k = \frac{1}{k}\,;$$

thus $\sup B_{k+1} < \inf B_k$ for each $k \in \mathbb{N}$. Therefore, B_{k+1} is on the left of B_k for each $k \in \mathbb{N}$. We also note that every element in A is an isolated point of A.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in A.

(a) Suppose that there exists $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} \,|\, x_n \in B_k\} = \infty$. Then $\lim\limits_{n\to\infty} x_n \in \overline{B_k}$.

(b) Suppose that for all $k \in \mathbb{N}$ we have $\{n \in \mathbb{N} \,|\, x_n \in B_k\} < \infty$. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying that $x_{n_{j+1}} < x_{n_j}$ for all $j \in \mathbb{N}$. Such a subsequence must converge to 0 since for each $k \in \mathbb{N}$ only finitely many terms of x_{n_j} belongs to the set $B_1 \cup B_2 \cup \cdots \cup B_k$ while the supremum of the rest of the subsequence is not greater than $\inf B_k$.

Therefore, by the fact that $\overline{B_k} = B_k \cup \{\frac{1}{k}\}$, we find that

$$\bar{A} = A \cup \left\{\frac{1}{k} \,\middle|\, k \in \mathbb{N}\right\} \cup \{0\}.$$

Then the fact that every point in A is an isolated point of A implies that

$$A' = \bar{A} \backslash \text{ collection of isolated point of } A = \left\{\frac{1}{k} \,\middle|\, k \in \mathbb{N}\right\} \cup \{0\}.$$

Noting that every point of A' except $\{0\}$ is an isolated point of A', we have $A^{(2)} = \{0\}$ so that $A^{(3)} = \varnothing$.

3. Following 5, we have a clear picture how to construct such a set. Let

$$A_m = \left\{\frac{1}{i_1} + \frac{1}{i_2} + \cdots + \frac{1}{i_m} \,\middle|\, i_j \in \mathbb{N} \text{ and } i_{j+1} - 1 > i_j(i_j - 1) \text{ for all } 1 \leqslant j \leqslant m\right\}.$$

Then $A_m' = A_{m-1} \cup \{0\}$, $A_m^{(2)} = A_{m-2} \cup \{0\}$, $\cdots$, $A_m^{(k)} = A_{m-k} \cup \{0\}$ if $m > k$, $A_m^{(m)} = \{0\}$ and $A_m^{(m+1)} = \varnothing$. □

Problem 4. Recall that a cluster point x of a sequence $\{x_n\}_{n=1}^\infty$ satisfies that

$$\forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(x, \varepsilon)\big\} = \infty.$$

Show that the collection of cluster points of a sequence (in a metric space) is closed.

Proof. Let (M, d) be a metric space, $\{x_k\}_{k=1}^\infty$ be a sequence in M, and A be the collection of cluster points of $\{x_k\}_{k=1}^\infty$. We would like to show that $A \supseteq \bar{A}$.

Let $y \in A^\complement$. Then y is not a cluster point of $\{x_k\}_{k=1}^\infty$; thus

$$\exists\, \varepsilon > 0 \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(y, \varepsilon)\big\} < \infty.$$

For $z \in B(y, \varepsilon)$, let $r = \varepsilon - d(y, z) > 0$. Then $B(z, r) \subseteq B(y, \varepsilon)$ (see Figure 1 or check rigorously using the triangle inequality). As a consequence, $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in B(z, r)\big\} < \infty$ which implies that $z \notin A$.

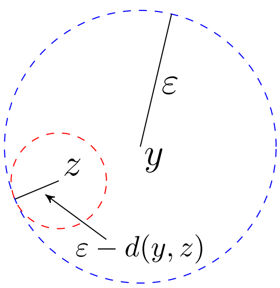

Figure 1: $B(z, \varepsilon - d(y, z)) \subseteq B(y, \varepsilon)$ if $z \in B(y, \varepsilon)$

Therefore, if $z \in B(y, \varepsilon)$ then $z \in A^\complement$; thus $B(y, \varepsilon) \cap A = \varnothing$. We then conclude that if $y \in A^\complement$ then $y \notin \bar{A}$. □

Problem 5. Let $(\mathcal{V}, \|\cdot\|)$ ba a normed vector space, and C be a non-empty convex set in $\mathcal{V}$.

1. Show that $\bar{C}$ is convex.

2. Show that if $\boldsymbol{x} \in \mathring{C}$ and $\boldsymbol{y} \in \bar{C}$, then $(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$. This result is sometimes called the ***line segment principle***.

3. Show that $\mathring{C}$ is convex (you may need the conclusion in 2 to prove this).

4. Show that $\text{cl}(\mathring{C}) = \text{cl}(C)$.

5. Show that $\text{int}(\bar{C}) = \text{int}(C)$.

Hint: 2. Prove by contradiction.
3 and 4. Use the line segment principle.
5. Show that $\boldsymbol{x} \in \text{int}(\bar{C})$ can be written as $(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z}$ for some $\boldsymbol{y} \in \mathring{C}$ and $\boldsymbol{z} \in B(\boldsymbol{x}, \varepsilon) \subseteq \bar{C}$.

Proof. 1. Let $\boldsymbol{x}, \boldsymbol{y} \in \bar{C}$ and $0 \leqslant \lambda \leqslant 1$ be given. Then there exist sequences $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ and $\{\boldsymbol{y}_k\}_{k=1}^{\infty}$ in C such that $\boldsymbol{x}_k \to \boldsymbol{x}$ and $\boldsymbol{y}_k \to \boldsymbol{y}$ as $k \to \infty$. Since C is convex, $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \in C$ for each $k \in \mathbb{N}$; thus by the fact that $C \subseteq \bar{C}$, $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \in \bar{C}$ for each $k \in \mathbb{N}$. Since $(1-\lambda)\boldsymbol{x}_k + \lambda\boldsymbol{y}_k \to (1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}$ as $k \to \infty$ and $\bar{C}$ is closed, we must have $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \bar{C}$; thus $\bar{C}$ is convex if C is convex.

2. Suppose the contrary that there exists $\lambda \in (0,1)$ such that $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \notin \mathring{C}$. Then for each $k \in \mathbb{N}$, there exists $\boldsymbol{z}_k \notin C$ such that

$$\big\|(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} - \boldsymbol{z}_k\big\| < \frac{1}{k} \qquad \forall\, k \in \mathbb{N}\,.$$

Since $\boldsymbol{y} \in \bar{C}$, there exists a sequence $\{\boldsymbol{y}_k\}_{k=1}^{\infty} \in C$ satisfying

$$\|\boldsymbol{y}_k - \boldsymbol{y}\| < \frac{1}{\lambda k} \qquad \forall\, k \in N\,.$$

Therefore, if $k \in N$,

$$\big\|(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}_k - \boldsymbol{z}_k\big\| \leqslant \big\|(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} - \boldsymbol{z}_k\big\| + \|\lambda(\boldsymbol{y} - \boldsymbol{y}_k)\| < \frac{2}{k}\,;$$

thus

$$\Big\|\boldsymbol{x} - \frac{\boldsymbol{z}_k - \lambda\boldsymbol{y}_k}{1-\lambda}\Big\| < \frac{2}{k(1-\lambda)} \qquad \forall\, k \in \mathbb{N}\,.$$

Since $\boldsymbol{x} \in \mathring{C}$, there exists $N > 0$ such that $B\big(\boldsymbol{x}, \dfrac{2}{(1-\lambda)N}\big) \subseteq C$; thus $\dfrac{\boldsymbol{z}_k - \lambda\boldsymbol{y}_k}{1-\lambda} \in C$ whenever $k \geqslant N$. By the convexity of C,

$$\boldsymbol{z}_k = (1-\lambda)\frac{\boldsymbol{z}_k - \lambda\boldsymbol{y}_k}{1-\lambda} + \lambda\boldsymbol{y}_k \in C\,,$$

a contradiction.

3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathring{C}$. By the line segment principle, $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$ (since $\mathring{C} \subseteq \bar{C}$ so that $y \in \bar{C}$). This further implies that $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in [0,1]$ since $\boldsymbol{x}, \boldsymbol{y} \in \mathring{C}$; thus $\mathring{C}$ is convex.

4. It suffices to show that $\mathrm{cl}(\mathring{C}) \supseteq \mathrm{cl}(C)$. Let $\boldsymbol{x} \in \mathrm{cl}(C)$. Pick any $\boldsymbol{y} \in \mathring{C}$. By the line segment principle,
$$\boldsymbol{x}_k \equiv \big(1 - \frac{1}{k}\big)\boldsymbol{x} + \frac{1}{k}\boldsymbol{y} \in \mathring{C} \qquad \forall\, k \geqslant 2\,.$$
Since $\boldsymbol{x}_k \to \boldsymbol{x}$ as $k \to \infty$, we find that $\boldsymbol{x} \in \mathrm{cl}(\mathring{C})$.

5. It suffices to show that $\mathrm{int}(\bar{C}) \subseteq \mathrm{int}(C)$. Let $\boldsymbol{x} \in \mathrm{int}(\bar{C})$. Then there exists $\varepsilon > 0$ such that $B(\boldsymbol{x}, \varepsilon) \subseteq \bar{C}$. Let $\boldsymbol{y} \in \mathrm{int}(C)$. If $\boldsymbol{y} = \boldsymbol{x}$, then $\boldsymbol{x} \in \mathrm{int}(C)$. If $\boldsymbol{y} \neq \boldsymbol{x}$, define $\boldsymbol{z} = \boldsymbol{x} + \alpha(\boldsymbol{x} - \boldsymbol{y})$, where
$$\alpha = \frac{\varepsilon}{2\|\boldsymbol{x} - \boldsymbol{y}\|}\,.$$
Then $\|\boldsymbol{x} - \boldsymbol{z}\| = \dfrac{\varepsilon}{2}$; thus $\boldsymbol{z} \in B(\boldsymbol{x}, \varepsilon)$ which further implies that $\boldsymbol{z} \in \bar{C}$. By the line segment principle implies that $(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z} \in \mathring{C}$ for all $\lambda \in (0,1)$. Taking $\lambda = \dfrac{1}{1+\alpha}$, we find that
$$(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z} = \frac{\alpha}{1+\alpha}\boldsymbol{y} + \frac{1}{1+\alpha}\big(\boldsymbol{x} + \alpha(\boldsymbol{x} - \boldsymbol{y})\big) = \boldsymbol{x}$$
which shows that $\boldsymbol{x} \in \mathrm{int}(C)$. □

Problem 6. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. Show that for all $\boldsymbol{x} \in \mathcal{V}$ and $r > 0$,
$$\mathrm{int}\big(B[\boldsymbol{x}, r]\big) = B(\boldsymbol{x}, r)\,.$$
Is the identity above true in general metric space?

Proof. Let $\boldsymbol{y} \in \mathcal{V}$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| = r$. Then $\boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x}) \in B[\boldsymbol{x}, r]^\complement$ for all $|\lambda| > 1$. In particular, $\boldsymbol{y}_n \equiv \boldsymbol{x} + \big(1 + \frac{1}{n}\big)(\boldsymbol{y} - \boldsymbol{x}) \in B[\boldsymbol{x}, r]^\complement$ for all $n \in \mathbb{N}$. Moreover,
$$\|\boldsymbol{y}_n - \boldsymbol{y}\| = \frac{1}{n}\|\boldsymbol{x} - \boldsymbol{y}\| = \frac{r}{n} \to 0 \quad \text{as} \quad n \to \infty\,.$$
Therefore, $\lim\limits_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}$ which implies that $\boldsymbol{y} \in \partial B[\boldsymbol{x}, r]$ (since $\boldsymbol{y} \in B[\boldsymbol{x}, r]$ and $\boldsymbol{y}$ is the limit of a sequence from $B[\boldsymbol{x}, r]^\complement$); thus
$$\big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x} - \boldsymbol{y}\| = r\big\} \subseteq \partial B[\boldsymbol{x}, r]\,.$$
On the other hand, $B(\boldsymbol{x}, r)$ is open and $B[\boldsymbol{x}, r] = B(\boldsymbol{x}, r) \cup \big\{\boldsymbol{y} \in \mathcal{V} \,\big|\, \|\boldsymbol{x} - \boldsymbol{y}\| = r\big\}$. Therefore, $B(x, r)$ is the largest open set contained inside $B[\boldsymbol{x}, r]$; thus $B(\boldsymbol{x}, r) = \mathrm{int}(B[\boldsymbol{x}, r])$.

The identity is not true in general metric space. For example, consider the metric space (M, d_0), where d_0 is the discrete metric on set M. For each $x \in M$, $B(x, 1) = \{x\}$ but $B[x, 1] = M$. Since M is open, $\mathrm{int}(M) = M$; thus $\mathrm{int}\big(B[x, 1]\big) \neq B(x, 1)$ as long as $\#M > 1$. □

Problem 7. Let $\mathcal{M}_{n\times n}$ denote the collection of all $n\times n$ square real matrices, and $(\mathcal{M}_{n\times n}, \|\cdot\|_{p,q})$ be a normed space with norm $\|\cdot\|_{p,q}$ given in Problem 6 of Exercise 5. Show that the set

$$\mathrm{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is an open set in $\mathcal{M}_{n\times n}$. The set $\mathrm{GL}(n)$ is called the general linear group.

Proof. Let $A \in \mathrm{GL}(n)$ be given. Then $A^{-1} \in \mathcal{M}_{n\times n}$ exists; thus

$$\|A^{-1}\boldsymbol{x}\|_2 \leqslant \|A^{-1}\|_{2,2}\|\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n$$

which, using the fact that $A : \mathbb{R}^n \xrightarrow[\text{onto}]{1-1} \mathbb{R}^n$, implies that

$$\frac{1}{\|A^{-1}\|_{2,2}}\|\boldsymbol{x}\|_2 \leqslant \|A\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Let $r = \dfrac{1}{\|A^{-1}\|_{2,2}}$. For $B \in B(A,r)$, we have $\|A-B\|_{2,2} < r$; thus for each $\boldsymbol{x} \in \mathbb{R}^n$,

$$r\|\boldsymbol{x}\|_2 = \frac{1}{\|A^{-1}\|_{2,2}}\|\boldsymbol{x}\|_2 \leqslant \|A\boldsymbol{x}\|_{\mathbb{R}^n} \leqslant \|(A-B)\boldsymbol{x}\|_2 + \|B\boldsymbol{x}\|_2 \leqslant \|A-B\|_{2,2}\|\boldsymbol{x}\|_{\mathbb{R}^n} + \|B\boldsymbol{x}\|_2$$

which further implies that

$$\|B\boldsymbol{x}\|_2 \geqslant \big(r - \|A-B\|_{2,2}\big)\|\boldsymbol{x}\|_2 \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n\,.$$

Therefore, $B\boldsymbol{x} = \boldsymbol{0}$ if and only if $\boldsymbol{x} = \boldsymbol{0}$ which shows that B is invertible; thus we established that

$$\text{for each } A \in \mathrm{GL}(n), \text{ there exists } r = \frac{1}{\|A^{-1}\|_{2,2}} > 0 \text{ such that } B(A,r) \subseteq \mathrm{GL}(n).$$

This shows that $\mathrm{GL}(n)$ is open. □

Problem 8. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $U \subseteq \mathbb{R}$ is open, then

$$U = \bigcup_{k\in\mathcal{I}} (a_k, b_k)\,,$$

where $\mathcal{I}$ is countable, and $(a_k,b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.

Hint: For each point $x \in U$, define $L_x = \big\{y \in \mathbb{R} \,\big|\, (y,x) \subseteq U\big\}$ and $R_x = \big\{y \in \mathbb{R} \,\big|\, (x,y) \subseteq U\big\}$. Define $I_x = (\inf L_x, \sup R_x)$. Show that $I_x = I_y$ if $(x,y) \in U$ and if $(x,y) \nsubseteq U$ then $I_x \cap I_y = \varnothing$

Proof. As suggested in the hint, for each point $x \in U$ we define $L_x = \big\{y \in \mathbb{R} \,\big|\, (y,x) \subseteq U\big\}$ and $R_x = \big\{y \in \mathbb{R} \,\big|\, (x,y) \subseteq U\big\}$. We note that $a \equiv \inf L_x \notin U$ since if $a \in U$, by the openness of U there exists $r > 0$ such that $(a-r, a+r) \subseteq U$ which implies that $(a-r, x) \subseteq U$ so that $a - r \in L_x$, a contradiction to the fact that $a = \inf L_x$. Similarly, $\sup R_x \notin U$. Therefore, $I_x = (\inf L_x, \sup L_x)$ is the maximal connected subset of U containing x.

If $x, y \in U$ and $(x,y) \subseteq U$, then $(L_x, y) = (L_x, x) \cup \{x\} \cup (x,y) \subseteq U$ which implies that $L_x \subseteq L_y$. On the other hand, if $z \in L_y$, then $z \leqslant x$ and $(z,x) \subseteq U$; thus $L_y \subseteq L_x$ which implies that $L_x = L_y$ if $x, y \in U$ and $(x,y) \subseteq U$. This shows that $I_x = I_y$ if $x, y \in U$ and $(x,y) \subseteq U$. Moreover, if $x, y \in U$ but $(x,y) \nsubseteq U$, then there exists $x < z < y$ such that $z \notin U$; thus $\sup R_x \leqslant z \leqslant \inf L_y$ which implies that $I_x \cap I_y = \varnothing$. Therefore, we establish that

1. if $x, y \in U$ and $(x, y) \subseteq U$, then $I_x = I_y$.

2. if $x, y \in U$ and $(x, y) \nsubseteq U$, then $I_x \cap I_y = \varnothing$.

This implies that U is the union of disjoint open intervals. Since every such open interval contains a rational number, we can denote each such open interval as I_k, where k belongs to a countable index set $\mathcal{I}$. Write $I_k = (a_k, b_k)$, then $U = \bigcup\limits_{k \in \mathcal{I}} (a_k, b_k)$. ▫

Problem 9. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be ***perfect*** if $A = A'$ (so that there is no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap\limits_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set.

2. Show that C is uncountable.

3. Find $\text{int}(C)$.

Proof. 1. Let $x \in C$. Then $x \in E_N$ for some $N \in \mathbb{N}$. For each $n \in \mathbb{N}$, E_n is the union of disjoint closed intervals with length $\frac{1}{3^n}$, and ∂E_n consists of the end-points of these disjoint closed intervals whose union is E_n. Therefore, there exists $x_n \in \partial E_{N+n-1} \backslash \{x\}$ such that $|x_n - x| < \frac{1}{3^{N-1+n}}$. Since $\partial E_n \subseteq C$ for each $n \in \mathbb{N}$, we find that $\{x_n\}_{n=1}^{\infty} \in C \backslash \{x\}$. Moreover, $\lim\limits_{n \to \infty} x_n = x$; thus $x \in C'$ which shows $C \subseteq C'$. Since C is the intersection of closed sets, C is closed; thus

$$C \subseteq C' \subseteq \bar{C} = C$$

so we establish that $C' = C$.

2. For $x \in [0, 1]$, write x in ternary expansion (三進位展開); that is,

$$x = 0.d_1 d_2 d_3 \cdots\cdots .$$

Here we note that repeated 2's are chosen by preference over terminating decimals. For example, we write $\frac{1}{3}$ as $0.02222\cdots$ instead of 0.1. Define

$$A = \{x = 0.d_1d_2d_3\cdots \mid d_j \in \{0,2\} \text{ for all } j \in \mathbb{N}\}.$$

Note each point in ∂E_n belongs to A; thus $A \subseteq C$. On the other hand, A has a one-to-one correspondence with $[0,1]$ $\big(x = 0.d_1d_2\cdots \in A \Leftrightarrow y = 0.\frac{d_1}{2}\frac{d_2}{2}\cdots \in [0,1]$, where y is expressed in binary expansion (二進位展開) with repeated 1's instead of terminating decimals$\big)$. Since $[0,1]$ is uncountable, A is uncountable; thus C is uncountable.

3. If $\text{int}(C)$ is non-empty, then by the fact that $\text{int}(C)$ is open in $(R, |\cdot|)$, by Problem 7 the Cantor set C contains at least one interval (x,y). Note that there exists $N > 0$ such that $|x-y| < \frac{1}{3^n}$ for all $n \geqslant N$. Since the length of each interval in E_n has length $\frac{1}{3^n}$, we find that if $n \geqslant N$, the interval (x,y) is not contained in any interval of E_n. In other words, there must be $z \in (x,y)$ such that $z \in E_n^{\complement}$ which shows that $(x,y) \nsubseteq \bigcap_{n=1}^{\infty} E_n$. Therefore, $\text{int}(C) = \varnothing$. □

Problem 10. Let $\mathcal{V}$ be a vector fields over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \subseteq \mathcal{V}$ is a basis for $\mathcal{V}$; that is, every $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as

$$\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n = \sum_{i=1}^{n} x^{(i)}\mathbf{e}_i.$$

Define $\|\boldsymbol{x}\|_2 = \Big(\sum_{i=1}^{n} |x^{(i)}|^2\Big)^{\frac{1}{2}}$.

1. Show that $\|\cdot\|_2$ is a norm on $\mathcal{V}$.

2. Show that K is compact in $(\mathcal{V}, \|\cdot\|_2)$ if and only if K is closed and bounded.

Proof. 1. By Cauchy-Schwarz inequality.

2. It suffices to show the "if" direction. Let $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ be a sequence in K. Write $\boldsymbol{x}_k = \sum_{i=1}^{n} x_k^{(i)}\mathbf{e}_i$. Since $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$\|\boldsymbol{x}_k\|_2 \leqslant M \qquad \forall\, k \in \mathbb{N}.$$

Therefore, $|x_k^{(i)}| \leqslant M$ for all $k \in \mathbb{N}$ and $1 \leqslant i \leqslant n$; thus for each $1 \leqslant i \leqslant n$, $\{x_k^{(i)}\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{F}$. By the Bolzano-Weierstrass Theorem (treat $\mathbb{C}$ as $\mathbb{R}^2$ to apply the theorem), there exists a subsequence $\{\boldsymbol{x}_{k_j}\}_{j=1}^{\infty}$ such that $\{x_{k_j}^{(i)}\}_{j=1}^{\infty}$ converges to some $x^{(i)} \in \mathbb{F}$. Let $\boldsymbol{x} = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$. Then

$$\|\boldsymbol{x}_{k_j} - \boldsymbol{x}\|_2 = \Big(\sum_{i=1}^{n} |x_{k_j}^{(i)} - x^{(i)}|^2\Big)^{\frac{1}{2}} \to 0 \text{ as } j \to \infty;$$

thus the closedness of K implies that $\boldsymbol{x} \in K$. □

Problem 11. Let (M, d) be a metric space.

1. Show that a closed subset of a compact set is compact.
2. Show that the union of a finite number of sequentially compact subsets of M is compact.
3. Show that the intersection of an arbitrary collection of sequentially compact subsets of M is sequentially compact.

Proof. 1. Let K be a compact set in M, F be a closed subset of K, and $\{x_k\}_{k=1}^{\infty}$ be a sequence in F. Then $\{x_k\}_{k=1}^{\infty}$ is a sequence in K; thus the sequential compactness of K implies that there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ with limit $x \in K$. Note that $\{x_{k_j}\}_{j=1}^{\infty}$ itself is a convergent sequence in F; thus the limit x of $\{x_{k_j}\}_{j=1}^{\infty}$ belongs to F by the closedness of F.

2. Let $K_1, K_2, \cdots, K_N$ be compact sets, and $K = \bigcup_{\ell=1}^{N} K_\ell$, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. Then there exists $1 \leqslant \ell_0 \leqslant N$ such that
$$\#\big\{n \in \mathbb{N} \,\big|\, x_n \in K_{\ell_0}\big\} = \infty\,.$$
Let $\{x_{n_k}\}_{k=1}^{\infty} \subseteq K_{\ell_0}$. By the compactness of K_{ℓ_0}, there exists a convergent subsequence $\big\{x_{n_{k_j}}\big\}_{j=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x \in K_{\ell_0} \subseteq K$. Since $\big\{x_{n_{k_j}}\big\}_{j=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, we conclude that every sequence in K has a convergent subsequence with limit in K; thus K is compact.

3. Since every compact set is closed, the intersection of an arbitrary collection of compact sets of M is closed. By 1, this intersection is also compact since the intersection is a closed set of any compact set (in the family). ▫

Problem 12. Given $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ a bounded sequence, define
$$A = \Big\{x \in \mathbb{R} \,\Big|\, \text{there exists a subsequence } \big\{a_{k_j}\big\}_{j=1}^{\infty} \text{ such that } \lim_{j\to\infty} a_{k_j} = x\Big\}\,.$$
Show that A is a non-empty sequentially compact set in $\mathbb{R}$. Furthermore, $\limsup\limits_{k\to\infty} a_k = \sup A$ and $\liminf\limits_{k\to\infty} a_k = \inf A$.

Proof. Note that A is the collection of cluster points of bounded sequence $\{a_k\}_{k=1}^{\infty}$; thus Problem 3 of Exercise 7 shows that A is closed. Moreover, A is bounded since $\{a_k\}_{k=1}^{\infty}$ is bounded; thus $\sup A \in A$ and $\inf A \in A$. The desired result then follows from the fact that $\limsup\limits_{k\to\infty} a_k$ is the largest cluster point of $\{a_k\}_{k=1}^{\infty}$ and $\liminf\limits_{k\to\infty} a_k$ is the least cluster point of $\{a_k\}_{k=1}^{\infty}$; thus $\limsup\limits_{k\to\infty} a_k = \sup A \in A$ and $\liminf\limits_{k\to\infty} a_k = \inf A \in A$. ▫

Problem 13. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$
Problem 12 of Exercise 5 shows that d is a metric on $\mathbb{R}^2$. Consider the metric space $(\mathbb{R}^2, d)$.

1. Find $B(x,r)$ with $r<1$, $r=1$ and $r>1$.

2. Show that the set $\{c\}\times[a,b]\subseteq(\mathbb{R}^2,d)$ is closed and bounded.

3. Examine whether the set $\{c\}\times[a,b]\subseteq(\mathbb{R}^2,d)$ is sequentially compact or not.

Problem 14. Let ℓ^2 be the collection of all sequences $\{x_k\}_{k=1}^\infty\subseteq\mathbb{R}$ such that $\sum\limits_{k=1}^\infty |x_k|^2<\infty$. In other words,

$$\ell^2=\Big\{\{x_k\}_{k=1}^\infty\,\Big|\,x_k\in\mathbb{R}\text{ for all }k\in\mathbb{N},\ \sum_{k=1}^\infty|x_k|^2<\infty\Big\}.$$

Define $\|\cdot\|_2:\ell^2\to\mathbb{R}$ by

$$\big\|\{x_k\}_{k=1}^\infty\big\|_2=\Big(\sum_{k=1}^\infty|x_k|^2\Big)^{\frac{1}{2}}.$$

1. Show that $\|\cdot\|_2$ is a norm on ℓ^2. The normed space $(\ell^2,\|\cdot\|)$ usually is denoted by ℓ^2.

2. Show that $\|\cdot\|_2$ is induced by an inner product.

3. Show that $(\ell^2,\|\cdot\|_2)$ is complete.

4. Let $A=\big\{\boldsymbol{x}\in\ell^2\,\big|\,\|\boldsymbol{x}\|_2\leqslant 1\big\}$. Is A sequentially compact or not?

Proof. 1. Let $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ be elements in ℓ^2 and $c\in\mathbb{R}$. Clearly $\big\|\{x_k\}_{k=1}^\infty\big\|\geqslant 0$ and $\big\|\{x_k\}_{k=1}^\infty\big\|=0$ if and only if $x_k=0$ for all $k\in\mathbb{N}$. Moreover,

$$\big\|c\{x_k\}_{k=1}^\infty\big\|_2=\big\|\{cx_k\}_{k=1}^\infty\big\|_2=\Big(\sum_{k=1}^\infty|cx_k|^2\Big)^{\frac{1}{2}}=|c|\Big(\sum_{k=1}^\infty|x_k|^2\Big)^{\frac{1}{2}}=|c|\big\|\{x_k\}_{k=1}^\infty\big\|_2.$$

Finally, since the 2-norm for $\mathbb{R}^n$ is a norm, we must have

$$\Big(\sum_{k=1}^n|x_k+y_k|^2\Big)^{\frac{1}{2}}\leqslant\Big(\sum_{k=1}^n|y_k|^2\Big)^{\frac{1}{2}}+\Big(\sum_{k=1}^n|y_k|^2\Big)^{\frac{1}{2}}$$

Passing to the limit as $n\to\infty$, we find that

$$\begin{aligned}\big\|\{x_k\}_{k=1}^\infty+\{y_k\}_{k=1}^\infty\big\|&=\big\|\{x_k+y_k\}_{k=1}^\infty\big\|_2=\lim_{n\to\infty}\Big(\sum_{k=1}^n|x_k+y_k|^2\Big)^{\frac{1}{2}}\\&\leqslant\lim_{n\to\infty}\Big[\Big(\sum_{k=1}^n|y_k|^2\Big)^{\frac{1}{2}}+\Big(\sum_{k=1}^n|y_k|^2\Big)^{\frac{1}{2}}\Big]=\big\|\{x_k\}_{k=1}^\infty\|_2+\big\|\{y_k\}_{k=1}^\infty\|_2.\end{aligned}$$

Therefore, the triangle inequality for $\|\cdot\|_2$ holds.

2. The norm $\|\cdot\|_2$ is indeed the norm induced by the inner product

$$\big\langle\{x_k\}_{k=1}^\infty,\{y_k\}_{k=1}^\infty\big\rangle=\sum_{k=1}^\infty x_ky_k\qquad\{x_k\}_{k=1}^\infty,\{y_k\}_{k=1}^\infty\in\ell^2.$$

3. Let $\{\boldsymbol{x}_k\}_{k=1}^\infty$ be a Cauchy sequence. Write $\boldsymbol{x}_k = \big\{x_\ell^{(k)}\big\}_{\ell=1}^\infty$. Then for each $\ell \in \mathbb{N}$ the sequence $\{x_\ell^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. In fact, for a given $\varepsilon > 0$, there exists $N > 0$ such that

$$\|\boldsymbol{x}_m - \boldsymbol{x}_n\|_2 < \varepsilon \qquad \text{whenever} \quad m, n \geqslant N$$

which implies that for each $\ell \in \mathbb{N}$,

$$\big|x_\ell^{(m)} - x_\ell^{(n)}\big| \leqslant \|\boldsymbol{x}_m - \boldsymbol{x}_n\|_2 < \varepsilon \qquad \text{whenever} \quad m, n \geqslant N\,.$$

By the completeness of $\mathbb{R}$, $\lim\limits_{k\to\infty} x_\ell^{(k)} = x_\ell$ exists for each $\ell \in \mathbb{N}$. Define $\boldsymbol{x} = \{x_\ell\}_{\ell=1}^\infty$.

Claim: $\boldsymbol{x} \in \ell^2$.

Proof of claim: By Proposition 2.58 in the lecture note, every Cauchy sequence is bounded; thus there exists $M > 0$ such that $\|\boldsymbol{x}_k\|_2 \leqslant M$ for all $k \in \mathbb{N}$. This implies that

$$\sum_{\ell=1}^n \big|x_\ell^{(k)}\big|^2 \leqslant M^2 \qquad \forall\, k, n \in \mathbb{N}\,;$$

thus

$$\sum_{\ell=1}^n |x_\ell|^2 = \sum_{\ell=1}^n \lim_{k\to\infty} \big|x_\ell^{(k)}\big|^2 = \lim_{k\to\infty} \sum_{\ell=1}^n \big|x_\ell^{(k)}\big|^2 \leqslant M^2 \qquad \forall\, n \in \mathbb{N}\,.$$

Therefore, $\|\boldsymbol{x}\|^2 = \sum\limits_{\ell=1}^\infty |x_\ell|^2 \leqslant M^2$ which implies that $\boldsymbol{x} \in \ell^2$. □

Next we show that $\{\boldsymbol{x}_k\}_{k=1}^\infty$ converges to $\boldsymbol{x}$ (in ℓ^2). Let $\varepsilon > 0$ be given. Since $\{\boldsymbol{x}_k\}_{k=1}^\infty$ is a Cauchy sequence, there exists $N > 0$ such that

$$\|\boldsymbol{x}_m - \boldsymbol{x}_n\|_2 < \frac{\varepsilon}{2} \qquad \text{whenever} \quad n, m \geqslant N\,.$$

Then similar to the proof of claim, for each $r \in \mathbb{N}$ and $n \geqslant N$ we have

$$\sum_{\ell=1}^r |x_\ell^{(n)} - x_\ell|^2 = \sum_{\ell=1}^r \lim_{m\to\infty} |x_\ell^{(n)} - x_\ell^{(m)}|^2 = \lim_{m\to\infty} \sum_{\ell=1}^r |x_\ell^{(n)} - x_\ell^{(m)}|^2 \leqslant \lim_{m\to\infty} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2^2 \leqslant \frac{\varepsilon^2}{4}\,;$$

thus if $n \geqslant N$,

$$\|\boldsymbol{x}_n - \boldsymbol{x}\|_2^2 = \sum_{\ell=1}^\infty |x_\ell^{(n)} - x_\ell|^2 \leqslant \frac{\varepsilon^2}{4} < \varepsilon\,.$$

Therefore, $\{\boldsymbol{x}_n\}_{n=1}^\infty$ converges to $\boldsymbol{x}$ so that we established that every Cauchy sequence in $(\ell^2, \|\cdot\|_2)$ converges to a point in ℓ^2. This shows that $(\ell^2, \|\cdot\|_2)$ is complete.

4. Consider the sequence $\{\boldsymbol{x}_k\}_{k=1}^\infty$ in ℓ^2 given by that $\boldsymbol{x}_k = \big\{x_\ell^{(k)}\big\}_{\ell=1}^\infty$ with $x_\ell^{(k)} = \delta_{k\ell}$, where $\delta_{k\ell}$ is the Kronecker delta. Then $\|\boldsymbol{x}_k\|_2 = 1$ for all $k \in \mathbb{N}$. On the other hand, if a subsequence of $\{x_k\}_{k=1}^\infty$ converges, it must converge to the zero sequence (since $x_\ell^{(k)} = 0$ for all ℓ except $\ell = k$) so that $\lim\limits_{j\to\infty} \big\|\boldsymbol{x}_k\big\|_2 = 0$, a contradiction. □

Problem 15. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A,B) = \inf\left\{\|x-y\|_2 \,\middle|\, x \in A, y \in B\right\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A,B)$ as $d(x,B)$ (which is consistent with the one given in Proposition 3.6 of the lecture note).

(1) Prove that $d(A,B) = \inf\left\{d(x,B) \,\middle|\, x \in A\right\}$.

(2) Show that $\left|d(x_1,B) - d(x_2,B)\right| \leqslant \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \left\{x \in \mathbb{R}^n \,\middle|\, d(x,B) < \varepsilon\right\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap\limits_{\varepsilon>0} B_\varepsilon = \mathrm{cl}(B)$.

(4) If A is sequentially compact, show that there exists $x \in A$ such that $d(A,B) = d(x,B)$.

(5) If A is closed and B is sequentially compact, show that there exists $x \in A$ and $y \in B$ such that $d(A,B) = d(x,y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

Proof. The proof of (1)-(4) does not rely on the structure of $(\mathbb{R}^n, \|\cdot\|_2)$, so in the proofs of (1)-(4) we write $d(\boldsymbol{x},\boldsymbol{y})$ instead of $\|\boldsymbol{x}-\boldsymbol{y}\|$.

(1) Define $f : A \times B \to \mathbb{R}$ by $f(\boldsymbol{a},\boldsymbol{b}) = d(\boldsymbol{a},\boldsymbol{b})$. By Problem **??**,

$$\inf_{(\boldsymbol{a},\boldsymbol{b})\in A\times B} f(\boldsymbol{a},\boldsymbol{b}) = \inf_{\boldsymbol{a}\in A}\Big(\inf_{\boldsymbol{b}\in B} f(\boldsymbol{a},\boldsymbol{b})\Big) = \inf_{\boldsymbol{b}\in B}\Big(\inf_{\boldsymbol{a}\in A} f(\boldsymbol{a},\boldsymbol{b})\Big).$$

Since $\inf\limits_{\boldsymbol{b}\in B} f(\boldsymbol{a},\boldsymbol{b}) = d(\boldsymbol{a},B)$, we conclude that

$$d(A,B) = \inf_{(\boldsymbol{a},\boldsymbol{b})\in A\times B} f(\boldsymbol{a},\boldsymbol{b}) = \inf_{\boldsymbol{a}\in A} d(\boldsymbol{a},B).$$

(2) Let $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. By the definition of infimum, there exists $\boldsymbol{z} \in B$ such that

$$d(\boldsymbol{x},B) \leqslant d(\boldsymbol{x},\boldsymbol{z}) < d(\boldsymbol{x},B) + \varepsilon.$$

By the definition of $d(\boldsymbol{y},B)$ and the triangle inequality,

$$d(\boldsymbol{y},B) \leqslant d(\boldsymbol{y},\boldsymbol{z}) \leqslant d(\boldsymbol{y},\boldsymbol{x}) + d(\boldsymbol{x},\boldsymbol{z}) < d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{x},B) + \varepsilon;$$

thus

$$d(\boldsymbol{y},B) - d(\boldsymbol{x},B) < d(\boldsymbol{x},\boldsymbol{y}) + \varepsilon.$$

A symmetric argument (switching $\boldsymbol{x}$ and $\boldsymbol{y}$) also shows that $d(\boldsymbol{x},B) - d(\boldsymbol{y},B) < d(\boldsymbol{x},\boldsymbol{y}) + \varepsilon$. Therefore,

$$\left|d(\boldsymbol{x},B) - d(\boldsymbol{y},B)\right| < d(\boldsymbol{x},\boldsymbol{y}) + \varepsilon.$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that

$$\left|d(\boldsymbol{x},B) - d(\boldsymbol{y},B)\right| \leqslant d(\boldsymbol{x},\boldsymbol{y}).$$

(3) Let $\boldsymbol{x} \in B_\varepsilon$. Define $r = \varepsilon - d(\boldsymbol{x}, B)$. Then $r > 0$; thus there exists $\boldsymbol{z} \in B$ such that

$$d(\boldsymbol{x}, B) \leqslant d(\boldsymbol{x}, \boldsymbol{z}) < d(\boldsymbol{x}, B) + \frac{r}{2} = \varepsilon\,.$$

Therefore, if $\boldsymbol{y} \in B\big(\boldsymbol{x}, \frac{r}{2}\big)$, then

$$d(\boldsymbol{y}, \boldsymbol{z}) \leqslant d(\boldsymbol{y}, \boldsymbol{x}) + d(\boldsymbol{x}, \boldsymbol{z}) < \frac{r}{2} + d(\boldsymbol{x}, B) + \frac{r}{2} = d(\boldsymbol{x}, B) + r = \varepsilon$$

which shows that $B\big(\boldsymbol{x}, \frac{r}{2}\big) \subseteq B_\varepsilon$. Therefore, B_ε is open.

Next, we note that

$$d(\boldsymbol{x}, B) = 0 \;\Leftrightarrow\; (\forall\, \varepsilon > 0)(d(\boldsymbol{x}, B) < \varepsilon) \;\Leftrightarrow\; (\forall\, \varepsilon > 0)(\boldsymbol{x} \in B_\varepsilon) \;\Leftrightarrow\; \boldsymbol{x} \in \bigcap_{\varepsilon>0} B_\varepsilon\,;$$

thus $d(\boldsymbol{x}, B) = 0$ if and only if $\boldsymbol{x} \in \bigcap\limits_{\varepsilon>0} B_\varepsilon$. By Proposition **??**, we conclude that $\bigcap\limits_{\varepsilon>0} B_\varepsilon = \bar{B}$.

(4) By the definition of infimum, for each $n \in \mathbb{N}$ there exists $\boldsymbol{a}_n \in A$ such that

$$d(A, B) \leqslant d(\boldsymbol{a}_n, B) < d(A, B) + \frac{1}{n}\,.$$

Since A is compact, there exists a convergent subsequence $\{\boldsymbol{a}_{n_j}\}_{j=1}^\infty$ of $\{\boldsymbol{a}_n\}_{n=1}^\infty$ with limit $\boldsymbol{a} \in A$. By the Sandwich Lemma,

$$d(\boldsymbol{a}_{n_j}, B) \to d(A, B) \quad \text{as} \quad j \to \infty\,.$$

On the other hand, (2) implies that

$$\big|d(\boldsymbol{a}_{n_j}, B) - d(\boldsymbol{a}, B)\big| \leqslant d(\boldsymbol{a}_{n_j}, \boldsymbol{a}) \to 0 \quad \text{as} \quad j \to \infty\,.$$

Therefore,

$$\big|d(\boldsymbol{a}, B) - d(A, B)\big| \leqslant \big|d(\boldsymbol{a}, B) - d(\boldsymbol{a}_{n_j}, B)\big| + \big|d(\boldsymbol{a}_{n_j}, B) - d(A, B)\big| \to 0 \quad \text{as} \quad j \to \infty$$

which establishes the existence of $\boldsymbol{a} \in A$ such that $d(\boldsymbol{a}, B) = d(A, B)$ if A is compact.

(5) By (4), there exists $\boldsymbol{b} \in B$ such that $d(A, B) = d(\boldsymbol{b}, A)$. Let $C = B[\boldsymbol{b}, d(A, B) + 1] \cap A$. Then

$$d(\boldsymbol{b}, A) = d(\boldsymbol{b}, C)$$

since every point $\boldsymbol{x} \in A \backslash C$ satisfies that $d(\boldsymbol{b}, \boldsymbol{x}) > d(A, B) + 1$. On the other hand, the Heine-Borel Theorem implies that C is compact; thus (4) implies that there exists $\boldsymbol{c} \in C$ such that $d(\boldsymbol{b}, C) = d(\boldsymbol{b}, \boldsymbol{c}) = \|\boldsymbol{b} - \boldsymbol{c}\|$. The desired result then follows from the fact that C is a subset of A (so that $\boldsymbol{c} \in A$).

(6) Let $A = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, xy \geqslant 1, x > 0\big\}$ and $B = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, xy \leqslant -1, x < 0\big\}$. Then A and B are closed set since they contain their boundaries. However, since $\boldsymbol{a} = \big(\frac{1}{n}, n\big) \in A$ and $\boldsymbol{b} = \big(-\frac{1}{n}, n\big) \in B$ for all $n \in \mathbb{N}$, $d(A, B) \leqslant d(\boldsymbol{a}, \boldsymbol{b}) = \frac{2}{n}$ for all $n \in \mathbb{N}$ which shows that $d(A, B) = 0$. However, the fact that $A \cap B = \varnothing$ implies that $d(\boldsymbol{a}, \boldsymbol{b}) > 0$ for all $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$. Therefore, in this case there are no $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$ such that $d(A, B) = d(\boldsymbol{a}, \boldsymbol{b})$. □