Exercise Problem Sets 7

Oct. 28. 2022

Problem 1. Let $\{a_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be two sequences of real numbers, and $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^\infty$ converges if $\sum\limits_{n=1}^\infty a_n$ converges.

Proof. First we note that if $n > m$,

$$\begin{aligned} |x_n - x_m| &= \big|x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m\big| \\ &\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &\leqslant a_{n-1} + a_{n-2} + \cdots + a_m = \sum_{k=m}^{n-1} a_k \,. \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\sum\limits_{k=1}^\infty a_k$ converges, the Cauchy criterion implies that there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+p} a_k\Big| = \big|a_n + a_{n+1} + \cdots + a_{n+p}\big| < \varepsilon \quad \text{whenever} \quad n \geqslant N \text{ and } p \geqslant 0\,.$$

Therefore, if $n > m \geqslant N$, by the fact $a_k > 0$ for all $k \in \mathbb{N}$, we have

$$|x_n - x_m| \leqslant \sum_{k=m}^{n-1} a_k < \varepsilon\,.$$

This implies that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, $\{x_n\}_{n=1}^\infty$ converges. □

Problem 2. Let $\sum\limits_{k=1}^\infty a_k$ be a conditionally convergent series. Show that $\sum\limits_{k=1}^\infty \big[1 + \operatorname{sgn}(a_k)\big]a_k$ and $\sum\limits_{k=1}^\infty \big[1 - \operatorname{sgn}(a_k)\big]a_k$ both diverge. Here the sign function sgn is defined by

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Proof. **Claim**: Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences of real numbers. If $\{x_n\}_{n=1}^\infty$ converges and $\{y_n\}_{n=1}^\infty$ diverges, then $\{x_n \pm y_n\}_{n=1}^\infty$ diverges.

To see the claim, suppose the contrary that $\{x_n + y_n\}_{n=1}^\infty$ converges. Then Theorem 1.40 in the lecture note implies that $\{x_n + y_n - x_n\}_{n=1}^\infty$ converges, which contradicts the assumption that $\{y_n\}_{n=1}^\infty$ diverges. Similarly, $\{x_n - y_n\}_{n=1}^\infty$ also diverges.

Let $S_n = \sum\limits_{k=1}^n a_k$ and $T_n = \sum\limits_{k=1}^n |a_k|$. Then $\{S_n\}_{n=1}^\infty$ converges but $\{T_n\}_{n=1}^\infty$ diverges. Therefore, the claim above shows that $\{S_n \pm T_n\}_{n=1}^\infty$ diverges. By the fact that $|a| = \operatorname{sgn}(a)a$ for all $a \in \mathbb{R}$, we have

$$S_n \pm T_n = \sum_{k=1}^n (a_k \pm |a_k|) = \sum_{k=1}^n \big[1 \pm \operatorname{sgn}(a_k)\big]a_k$$

so we conclude the desired result. □

Problem 3. Let $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ ba a sequence. A series $\sum\limits_{k=1}^{\infty} b_k$ is said to be a rearrangement of the series $\sum\limits_{k=1}^{\infty} a_k$ if there exists a rearrangement π of $\mathbb{N}$; that is, $\pi : \mathbb{N} \to \mathbb{N}$ is bijective, such that $b_k = a_{\pi(k)}$.

1. Show that if $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of the series $\sum\limits_{k=1}^{\infty} a_k$ converges and has the value $\sum\limits_{k=1}^{\infty} a_k$.
2. Show that if $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent, then for each $r \in \mathbb{R}$, there exists a rearrangement $\sum\limits_{k=1}^{\infty} a_{\pi(k)}$ of the series $\sum\limits_{k=1}^{\infty} a_k$ such that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$.

Proof. 1. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is an absolutely convergent series with limit a, and $\pi : \mathbb{N} \to \mathbb{N}$ is a rearrangement of $\mathbb{N}$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$\Big|\sum_{k=1}^{n} a_k - a\Big| < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{k=n+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N\,.$$

Choose $K > 0$ such that $\pi(n) > N$ if $n \geqslant K$. In fact, $K = \max\big\{\pi^{-1}(1), \cdots, \pi^{-1}(N)\big\} + 1$ suffices the purpose. Then $K \geqslant N$ and if $n \geqslant K$, $\pi\big(\{1, 2, \cdots, n\}\big) \supseteq \{1, 2, \cdots, N\}$. Therefore, if $n \geqslant K$,

$$\Big|\sum_{k=1}^{n} a_{\pi(k)} - a\Big| \leqslant \Big|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{N} a_k\Big| + \Big|\sum_{k=1}^{N} a_k - a\Big| \leqslant \sum_{k=N+1}^{\infty} |a_k| + \frac{\varepsilon}{2} < \varepsilon$$

which implies that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = a$.

2. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent. Let $\{a_{k_j}\}_{j=1}^{\infty}$ denote the subsequence of $\{a_k\}_{k=1}^{\infty}$ so that $a_{k_j} \geqslant 0$ for all $j \in \mathbb{N}$ and $a_k < 0$ if $k \in \mathbb{N}\backslash\{k_1, k_2, \cdots\}$. In other words, $\{a_{p_j}\}_{j=1}^{\infty}$ is the maximal subsequence of $\{a_k\}_{k=1}^{\infty}$ with non-negative terms. Let $\{a_{n_j}\}_{j=1}^{\infty}$ be the maximal subsequence of $\{a_k\}_{k=1}^{\infty}$ with negative terms. Then

$$\sum_{j=1}^{\infty} a_{p_j} = \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} a_{n_j} = -\infty\,.$$

Let $r \in \mathbb{R}$ be given, and use the notation $\sum\limits_{j=1}^{0}$ to denote summing nothing. Define $k_0 = 0$. Choose $k_1 \in \mathbb{N}$ be the unique natural number so that $\sum\limits_{j=1}^{k_1-1} a_{p_j} < r$ but $\sum\limits_{j=1}^{k_1} a_{p_j} > r$. Since $\sum\limits_{j=1}^{\infty} a_{n_j} = -\infty$, there exists a unique $k_2 \in \mathbb{N}$ such that $\sum\limits_{j=1}^{k_1} a_{p_j} + \sum\limits_{j=1}^{k_2-1} a_{n_j} > r$ but $\sum\limits_{j=1}^{k_1} a_{p_j} + \sum\limits_{j=1}^{k_2} a_{n_j} < r$. We continue this process, and obtain a sequence $\{k_\ell\}_{\ell=0}^{\infty}$ such that for each $\ell \in \mathbb{N}$,

(a) $\sum\limits_{j=1}^{k_{2\ell-1}-1} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell-2}} a_{n_j} < r$. (b) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell-2}} a_{n_j} > r$.

(c) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell-1}} a_{n_j} > r.$ (d) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell}} a_{n_j} < r.$

We then obtain a permutation of $\{a_n\}_{n=1}^{\infty}$:

$$\underbrace{a_{p_1}, \cdots, a_{p_{k_1}}}_{k_1 \text{ "}\geqslant 0\text{" terms}}, \underbrace{a_{n_1}, \cdots, a_{n_{k_2}}}_{k_2 \text{ "}< 0\text{" terms}}, \underbrace{a_{p_{k_1+1}}, \cdots, a_{p_{k_3}}}_{k_3 \text{ "}\geqslant 0\text{" terms}}, \underbrace{a_{n_{k_2+1}}, \cdots, a_{n_{k_4}}}_{k_4 \text{ "}< 0\text{" terms}}, \cdots .$$

Denote the permutation above by $\{a_{\pi(n)}\}_{n=1}^{\infty}$; that is, $\pi(1) = p_1, \cdots, \pi(k_1) = p_{k_1}, \pi(k_1+1) = n_1, \cdots, \pi(k_1 + k_2) = n_{k_2}$, and so on. Next we show that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$.

Let $\varepsilon > 0$ be given, and define $S_n = \sum\limits_{k=1}^{n} a_{\pi(k)}$. Since $\sum\limits_{n=1}^{\infty} a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$; thus there exists $N > 0$ such that $|a_n| < \varepsilon$ for all $n \geqslant N$. By the construction of $\{k_\ell\}_{\ell=1}^{\infty}$,

$$|S_n - S_{n-1}| = |a_{\pi(n)}| < \varepsilon \quad \text{whenever} \quad n \geqslant k_1 + k_2 + \cdots + k_N .$$

This implies that $S_n \in (r - \varepsilon, r + \varepsilon)$ whenever $n \geqslant k_1 + k_2 + \cdots + k_N$. Therefore,

$$\Big|\sum_{k=1}^{n} a_{\pi(k)} - r\Big| < \varepsilon \quad \text{whenever} \quad n \geqslant k_1 + k_2 + \cdots + k_N$$

which shows that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$. □

Problem 4. Consider the function $f(x) = \sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$.

1. Find the domain of f.

2. Show that for each $\varepsilon > 0$ and $0 < \delta < \pi$, there exists $N > 0$ and N depends only on ε and δ but is independent of x, such that

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \qquad \forall\, n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta] .$$

Proof. Let $S_n(x) = \sum\limits_{k=1}^{n} \sin(kx)$.

1. (a) If $x = 2n\pi$ for some $n \in \mathbb{Z}$ (or $x = 0 \pmod{2\pi}$), then $S_n(x) = 0$ for all $n \in \mathbb{N}$; thus for each $x = 0 \pmod{2\pi}$, $\{S_n(x)\}_{n=1}^{\infty}$ is bounded by 1.

   (b) If $x \neq 2n\pi$ for all $n \in \mathbb{Z}$ (or $x \neq 0 \pmod{2\pi}$), then

$$\begin{aligned} 2\sin\frac{x}{2} S_n(x) &= \sum_{k=1}^{n} 2\sin\frac{x}{2}\sin(kx) = \sum_{k=1}^{n} \cos\big(k - \frac{1}{2}\big)x - \cos\big(k + \frac{1}{2}\big)x \\ &= \cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x \end{aligned}$$

   which implies that

$$\big|S_n(x)\big| \leqslant \Big|\frac{\cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x}{2\sin\frac{x}{2}}\Big| \leqslant \frac{1}{\big|\sin\frac{x}{2}\big|} \qquad \forall\, x \neq 0 \pmod{2\pi} .$$

In either cases, for each $x \in \mathbb{R}$ there exists $M = M(x) \in \mathbb{R}$ such that $|S_n(x)| \leqslant M$. Therefore, the Dirichlet test $\big($with $a_k = \sin(kx)$ and $p_k = \frac{1}{k}\big)$ implies that f is defined everywhere; thus the domain of f is $\mathbb{R}$.

2. We mimic the proof of the Dirichlet test. Let $\varepsilon > 0$ and $\delta \in (0, 2\pi)$ be given. Then $\csc\frac{\delta}{2} > 0$; thus the Archimedean property of $\mathbb{R}$ implies that there exists $N > \frac{2}{\varepsilon}\csc\frac{\delta}{2}$. If $n \geqslant N$, $p \geqslant 0$ and $x \in [\delta, 2\pi - \delta]$ (thus $x \neq 0 \pmod{2\pi}$), then

$$\begin{aligned}
\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| &= \Big|\sum_{k=n}^{n+p} \big[S_{k+1}(x) - S_k(x)\big]\frac{1}{k}\Big| \\
&= \Big|-S_n(x)\frac{1}{n} + S_{n+1}(x)\big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + S_{n+p}(x)\big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) \\
&\quad + S_{n+p+1}(x)\frac{1}{n+p}\Big| \\
&\leqslant \frac{1}{\big|\sin\frac{x}{2}\big|}\Big[\frac{1}{n} + \big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + \big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) + \frac{1}{n+p}\Big] \\
&= \frac{2}{n\big|\sin\frac{x}{2}\big|} < \frac{\sin\frac{\delta}{2}}{\big|\sin\frac{x}{2}\big|}\varepsilon\,.
\end{aligned}$$

Since $x \in [\delta, 2\pi - \delta]$, $\sin\frac{x}{2}$ attains its minimum at $x = \delta$ or $2\pi - \delta$; thus

$$0 < \sin\frac{\delta}{2} \leqslant \sin\frac{x}{2} \qquad \forall\, x \in [\delta, 2\pi - \delta]\,.$$

Therefore,

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \qquad \text{whenever} \quad n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta]\,. \qquad \square$$

In the exercise of Chapter 3, we first introduce the concepts of accumulation points, isolated points and derived set of a set as follows.

Definition 0.1. Let (M, d) be a normed vector space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A\backslash\{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.

2. A point $x \in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A\backslash\{x\}$ that converges to x.

3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Problem 5. Let (M, d) be a metric space, and A be a subset of M.

1. Show that the collection of all isolated points of A is $A\backslash A'$.

2. Show that $A' = \bar{A}\backslash(A\backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A}\backslash A' = A\backslash A'$.

Proof. 1. By the definition of isolated points of sets,

$$\begin{aligned} x \in A\backslash A' &\Leftrightarrow x \in A \text{ and } x \text{ is not an accumulation point of } A \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x,\varepsilon) \cap A\backslash\{x\} = \varnothing \\ &\Leftrightarrow x \in A \text{ and } \exists\, \varepsilon > 0 \ni B(x,\varepsilon) \cap A \subseteq \{x\} \\ &\Leftrightarrow \exists\, \varepsilon > 0 \ni B(x,\varepsilon) \cap A = \{x\}\,; \end{aligned}$$

thus x is an isolated point of A if and only if $x \in A\backslash A'$.

2. First we show that $\bar{A} = A \cup A'$. To see this, let $x \in \bar{A}\backslash A$. By the fact that $A = A\backslash\{x\}$, there exists $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{x\}$ such that $\lim\limits_{n\to\infty} x_n = x$. Therefore, $x \in A'$ which implies that

$$\bar{A}\backslash A \subseteq A' \subseteq \bar{A}\,,$$

where we use the fact that $\bar{A} \supseteq A'$ to conclude the last inclusion. The inclusion relation above then shows that

$$\bar{A} = A \cup \bar{A} = A \cup (\bar{A}\backslash A) \subseteq A \cup A' \subseteq A \cup \bar{A} = \bar{A}\,;$$

thus we establish that $\bar{A} = A \cup A'$. This identity further shows that

$$\bar{A} \cap A^{\complement} = (A \cup A') \cap A^{\complement} = A' \cap A^{\complement} \subseteq A\,.$$

Now, using the identity $A\backslash B = A \cap B^{\complement}$ we find that

$$\begin{aligned} \bar{A}\backslash(A\backslash A') &= \bar{A} \cap \big(A \cap (A')^{\complement}\big)^{\complement} = \bar{A} \cap (A^{\complement} \cup A') = (\bar{A} \cap A^{\complement}) \cup (\bar{A} \cap A') \\ &= (\bar{A} \cap A^{\complement}) \cup A' = A'\,. \end{aligned}$$

Moreover, using $\bar{A} = A \cup A'$ we also have

$$\bar{A}\backslash A' = (A \cup A') \cap (A')^{\complement} = A \cap (A')^{\complement} = A\backslash A'\,.$$

□

Problem 6. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
2. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
3. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Find examples of that $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Proof. 1. Since $\mathrm{cl}(A)$ is closed, by the definition of closed set we have $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $\mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup B)$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$; thus $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. On the other hand, if $x \in \mathrm{cl}(A \cup B)$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A \cup B$ such that $\lim\limits_{n\to\infty} x_n = x$. Since $A \cup B$ contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$, at least one of A and B contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$. W.L.O.G., suppose that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \infty$. Let

$$\big\{n \in \mathbb{N} \,\big|\, x_n \in A\big\} = \big\{n_k \in \mathbb{N} \,\big|\, n_k < n_{k+1}\big\}\,.$$

Then $\{x_{n_k}\}_{k=1}^{\infty} \in A$. Since $x_n \to x$ as $n \to \infty$, we must have $x_{n_k} \to x$ as $k \to \infty$; thus $x \in \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

3. Let $x \in \text{cl}(A \cap B)$. Then
$$(\forall \varepsilon > 0)(B(x,\varepsilon) \cap (A \cap B) \neq \varnothing)\,.$$
Therefore, by the fact that $B(x,\varepsilon) \cap A \subseteq B(x,\varepsilon) \cap (A \cap B)$ and $B(x,\varepsilon) \cap B \subseteq B(x,\varepsilon) \cap (A \cap B)$, we have
$$(\forall \varepsilon > 0)(B(x,\varepsilon) \cap A \neq \varnothing) \qquad \text{and} \qquad (\forall \varepsilon > 0)(B(x,\varepsilon) \cap B \neq \varnothing)\,.$$
This implies that $x \in \bar{A} \cap \bar{B}$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^\complement$, then $\text{cl}(A \cap B) = \varnothing$, while $\bar{A} = \bar{B} = \mathbb{R}$ which provides an example of $\text{cl}(A \cap B) \subsetneq \bar{A} \cap \bar{B}$. □

Problem 7. Let A and B be subsets of a metric space (M,d). Show that

1. $\text{int}(\text{int}(A)) = \text{int}(A)$.
2. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.
3. $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$. Find examples of that $\text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B)$.

Proof. 1. Since $\text{int}(A)$ is open, by the definition of open sets we have $\text{int}(\text{int}(A)) = \text{int}(A)$.

2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\text{int}(A \cap B) \subseteq \text{int}(A)$ and $\text{int}(A \cap B) \subseteq \text{int}(B)$; thus $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$. On the other hand, let $x \in \text{int}(A) \cap \text{int}(B)$. Then $x \in \text{int}(A)$ and $x \in \text{int}(B)$; thus there exist $r_1, r_0 > 0$ such that
$$B(x,r_1) \subseteq A \qquad \text{and} \qquad B(x,r_)) \subseteq B\,.$$
Let $r = \min\{r_1, r_2\}$. Then $r > 0$, and $B(x,r) \subseteq B(x,r_1)$ and $B(x,r) \subseteq B(x,r_2)$. Therefore, $B(x,r) \subseteq A$ and $B(x,r) \subseteq B$ which further implies that $B(x,r) \subseteq A \cap B$; thus $x \in \text{int}(A \cap B)$.

3. Let $x \in \mathring{A} \cup \mathring{B}$. Then $x \in \mathring{A}$ or $x \in \mathring{B}$; thus there exists $r > 0$ such that $B(x,r) \subseteq A$ or $B(x,r) \subseteq B$. Therefore, there exists $r > 0$ such that $B(x,r) \subseteq A \cup B$ which shows that $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$. Note that if $A = \mathbb{Q}$ and $B = \mathbb{Q}^\complement$, then $\text{int}(A \cup B) = \mathbb{R}$ while $\text{int}(A) = \text{int}(B) = \varnothing$; thus we obtain an example of $\text{int}(A \cup B) \supsetneq \text{int}(A) \cup \text{int}(B)$. □

Problem 8. Let (M,d) be a metric space, and A be a subset of M. Show that
$$\partial A = \big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big).$$

Proof. By the definition of the boundary, $\partial A = \bar{A} \cap \overline{A^\complement}$; thus
$$\begin{aligned}
\big(A \cap \text{cl}(M \backslash A)\big) \cup \big(\text{cl}(A) \backslash A\big) &= \big(A \cap \overline{A^\complement}\big) \cup \big(\bar{A} \cap A^\complement\big) \\
&= \big[A \cup \big(\bar{A} \cap A^\complement\big)\big] \cap \big[\overline{A^\complement} \cup \big(\bar{A} \cap A^\complement\big)\big] = \bar{A} \cap \big[\big(\overline{A^\complement} \cup \bar{A}\big) \cap \big(\overline{A^\complement} \cup A^\complement\big)\big] \\
&= \bar{A} \cap \big[\big(\overline{A^\complement} \cup \bar{A}\big) \cap \overline{A^\complement}\big] = \partial A \cap \big(\overline{A^\complement} \cup \bar{A}\big) = \partial A\,,
\end{aligned}$$
where the last equality follows from that $\partial A \subseteq \bar{A}$ and $\partial A \subseteq \overline{A^\complement}$. □

Problem 9. Recall that in a metric space (M,d), a subset A is said to be dense in S if subsets satisfy $A \subseteq S \subseteq \text{cl}(A)$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.

2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \mathrm{cl}(A \cap B)$.

Proof. 1. If A is dense in S and if S is dense in T, then $A \subseteq S \subseteq \bar{A}$ and $S \subseteq T \subseteq \bar{S}$. Since $S \subseteq \bar{A}$, we must have $\bar{S} \subseteq \bar{A}$; thus

$$A \subseteq S \subseteq T \subseteq \bar{S} \subseteq \bar{A}$$

which shows that A is dense in T.

2. Let $x \in B$. Since B is open, there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subseteq B \subseteq S$. On the other hand, $x \in S$ since B is a subset of S; thus the denseness of A in S implies that

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap A \neq \varnothing)\,.$$

Therefore, for a given $\varepsilon > 0$, if $\varepsilon \geqslant \varepsilon_0$, then

$$B(x, \varepsilon) \cap (A \cap B) \supseteq B(x, \varepsilon_0) \cap (A \cap B) = B(x, \varepsilon_0) \cap A \neq \varnothing)$$

while if $\varepsilon < \varepsilon_0$, then

$$B(x, \varepsilon) \cap (A \cap B) = B(x, \varepsilon) \cap A \neq \varnothing\,.$$

This implies that

$$(\forall\, \varepsilon > 0)(B(x, \varepsilon) \cap (A \cap B) \neq \varnothing)\,;$$

thus $x \in \mathrm{cl}(A \cap B)$. □

Problem 10. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.

2. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.

3. If $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.

4. $\partial(A \cap B) \subseteq \partial A \cup \partial B$. Find examples of the equalities do not hold.

5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Proof. 1. We note that if F is closed, then

$$\partial F = \bar{F} \cap \overline{F^{\complement}} = F \cap \overline{F^{\complement}} \subseteq F\,. \tag{$\diamond$}$$

Since ∂F is closed, we must have $\partial(\partial A) \subseteq \partial A$. Note that if $A = \mathbb{Q} \cap [0, 1]$, then $\partial A = [0, 1]$; thus $\partial(\partial A) = \{0, 1\} \subsetneq \partial A$. Finally we show that $\partial(\partial A) = \partial A$ if A is closed. Using ($\diamond$), it suffices to show that $\partial A \subseteq \partial(\partial A)$. Using 2 of Problem 6,

$$\begin{aligned}\partial(\partial A) &= \partial A \cap \mathrm{cl}((\partial A)^{\complement}) = \partial A \cap \mathrm{cl}(A^{\complement} \cup \overline{A^{\complement}}^{\complement}) = \partial A \cap \big(\overline{A^{\complement}} \cup \mathrm{cl}(\overline{A^{\complement}}^{\complement}\big) \\ &= \big(\partial A \cap \overline{A^{\complement}}\big) \cup \big(\partial A \cap \mathrm{cl}(\overline{A^{\complement}}^{\complement})\big) \supseteq \big(\partial A \cap \overline{A^{\complement}}\big) = \partial A\,.\end{aligned}$$

2. Using 2 and 3 of Problem 6,

$$\begin{aligned}\partial(A\cup B) &= \overline{A\cup B}\cap \mathrm{cl}\big((A\cup B)^{\complement}\big) = \big(\bar{A}\cup\bar{B}\big)\cap \mathrm{cl}(A^{\complement}\cap B^{\complement}) \subseteq \big(\bar{A}\cup\bar{B}\big)\cap\big(\overline{A^{\complement}}\cap\overline{B^{\complement}}\big)\\ &= \big(\bar{A}\cap\overline{A^{\complement}}\cap\overline{B^{\complement}}\big)\cup\big(\bar{B}\cap\overline{A^{\complement}}\cap\overline{B^{\complement}}\big)\subseteq\big(\bar{A}\cap\overline{A^{\complement}}\big)\cup\big(\bar{B}\cap\overline{B^{\complement}}\big)=\partial A\cup\partial B\,.\end{aligned}$$

On the other hand, since $\partial A = \bar{A}\backslash\mathring{A}$ and $\mathring{A}\subseteq A$, we have

$$\bar{A}\subseteq A\cup\partial A\subseteq \mathring{A}\cup(\bar{A}\backslash\mathring{A}) = \bar{A}$$

which implies that $A\cup\partial A=\bar{A}$. Therefore,

$$\partial A\subseteq\bar{A}\subseteq\overline{A\cup B}=A\cup B\cup\partial(A\cup B)$$

and similarly $\partial B\subseteq A\cup B\cup\partial(A\cup B)$. Therefore,

$$\partial A\cup\partial B\subseteq\partial(A\cup B)\cup A\cup B\,.$$

Note that if $A=[-1,0]\cup\big(\mathbb{Q}\cap[0,1]\big)$ and $B=[-1,0]\cup(\mathbb{Q}^{\complement}\cap[0,1])$, then $A\cup B=[-1,1]$, $\partial A=\partial B=\{-1\}\cup[0,1]$ which implies that

$$\partial(A\cup B)=\{-1,1\}\subsetneq\partial A\cup\partial B\subsetneq A\cup B=\partial(A\cup B)\cup A\cup B\,.$$

3. By 2, it suffices to shows that $\partial A\cup\partial B\subseteq\partial(A\cup B)$ if $\bar{A}\cap\bar{B}=\varnothing$. Let $x\in\partial A\cup\partial B$. W.L.O.G., assume that $x\in\partial A$. Then $x\in\bar{A}$; thus $x\notin\bar{B}$ which further implies that there exists $\varepsilon_0>0$ such that $B(x,\varepsilon_0)\cap B=\varnothing$ or equivalently, $B(x,\varepsilon_0)\subseteq B^{\complement}$. Therefore, for given $r>0$, if $r<\varepsilon_0$, then

$$B(x,r)\cap(A\cup B)\supseteq B(x,r)\cap A\neq\varnothing$$

and

$$B(x,r)\cap\big((A\cup B)^{\complement}\big)=B(x,r)\cap(A^{\complement}\cap B^{\complement})=B(x,r)\cap A^{\complement}\neq\varnothing$$

while if $r\geqslant\varepsilon_0$, then

$$B(x,r)\cap(A\cup B)\subseteq B(x,\varepsilon_0)\cap(A\cup B)\supseteq B(x,\varepsilon_0)\cap A\neq\varnothing$$

and

$$B(x,r)\cap\big((A\cup B)^{\complement}\big)\supseteq B(x,\varepsilon_0)\cap(A^{\complement}\cap B^{\complement})=B(x,\varepsilon_0)\cap A^{\complement}\neq\varnothing\,.$$

As a consequence, for each $r>0$,

$$B(x,r)\cap(A\cup B)\neq\varnothing\quad\text{and}\quad B(x,r)\cap(A\cup B)^{\complement}\,;$$

thus $x\in\overline{A\cup B}$ and $x\in\mathrm{cl}\big((A\cup B)^{\complement}\big)$ which implies that $x\in\partial(A\cup B)$.

4. Using 2 and 3 of Problem 6,

$$\begin{aligned}\partial(A\cap B)&=\overline{A\cap B}\cap\mathrm{cl}\big((A\cap B)^{\complement}\big)=\overline{A\cap B}\cap\mathrm{cl}(A^{\complement}\cup B^{\complement})\subseteq\big(\bar{A}\cap\bar{B}\big)\cap\big(\overline{A^{\complement}}\cup\overline{B^{\complement}}\big)\\&=\big[\big(\bar{A}\cap\bar{B}\big)\cap\overline{A^{\complement}}\big]\cup\big[\big(\bar{A}\cap\bar{B}\big)\cap\overline{B^{\complement}}\big]\subseteq\big(\bar{A}\cap\overline{A^{\complement}}\big)\cup\big(\bar{B}\cap\overline{B^{\complement}}\big)=\partial A\cup\partial B\,.\end{aligned}$$

Note that if $A=\mathbb{Q}$ and $B=\mathbb{Q}^{\complement}$, then $\partial A=\partial B=\mathbb{R}$ but

$$\partial(A\cap B)=\varnothing\subsetneq\mathbb{R}=\partial A\cap\partial B\,.$$

5. Since ∂A is closed, 1 implies that $\partial(\partial(\partial A))=\partial(\partial A)$. □