Exercise Problem Sets 6

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Problem 1. Let (M,d) be a metric space, and $\{x_n\}_{n=1}^\infty$ be a sequence in M. Show that $\{x_n\}_{n=1}^\infty$ is Cauchy if and only if

$$\forall\, \varepsilon > 0, \exists\, y \in M \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin B(y,\varepsilon)\big\} < \infty\,.$$

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given. Since $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence, there exists $N > 0$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{whenever} \quad n, m \geqslant N\,.$$

Let $y = x_N$. Then

$$\big\{n \in \mathbb{N} \,\big|\, x_n \notin B(y,\varepsilon)\big\} \subseteq \{1, 2, \cdots, N-1\}$$

which shows that $\#\big\{n \in \mathbb{N} \,\big|\, x_n \notin B(y,\varepsilon)\big\} < \infty$.

"$\Leftarrow$" Let $\varepsilon > 0$ be given. By assumption there exists $y \in M$ such that $\#\Big\{n \in \mathbb{N} \,\Big|\, x_n \notin B\big(y, \frac{\varepsilon}{2}\big)\Big\} < \infty\,;$ thus

$$N \equiv \max\Big\{n \in \mathbb{N} \,\Big|\, x_n \notin B\big(y, \frac{\varepsilon}{2}\big)\Big\} < \infty\,.$$

Note that if $n > N$, $x_n \in B\big(y, \frac{\varepsilon}{2}\big)$ which implies that

$$d(x_n, y) < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > N\,.$$

Therefore, if $n, m > N$, the triangle inequality implies that

$$d(x_n, x_m) \leqslant d(x_n, y) + d(y, x_m) < \varepsilon$$

which shows that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. □

Problem 2. Let (M,d) be a metric space, and $\{x_k\}_{k=1}^\infty$ be a sequence in M. Show that

1. If $\{x_k\}_{k=1}^\infty$ is convergent, then $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence.
2. If $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence and a subsequence of $\{x_k\}_{k=1}^\infty$ converges to x, then $\{x_k\}_{k=1}^\infty$ converges to x.
3. x is a cluster point of $\{x_k\}_{k=1}^\infty$ if and only if $\forall\, \varepsilon > 0$ and $N > 0$, $\exists\, k > N$ with $d(x_k, x) < \varepsilon$.
4. x is a cluster point of $\{x_k\}_{k=1}^\infty$ if and only if there is a subsequence converging to x.
5. $\{x_k\}_{k=1}^\infty$ converges to x if and only if every subsequence of $\{x_k\}_{k=1}^\infty$ converges to x.
6. $\{x_k\}_{k=1}^\infty$ converges to x if and only if every proper subsequence of $\{x_k\}_{k=1}^\infty$ has a further subsequence that converges to x.

Problem 3. Let $(\mathcal{V}, \|\cdot\|)$ be a norm space, $\{a_k\}_{k=1}^\infty$ be a sequence in $\mathcal{V}$, and $\{b_n\}_{n=1}^\infty$ be the Cesàro mean of $\{a_k\}_{k=1}^\infty$; that is, $b_n = \frac{1}{n}\sum\limits_{k=1}^n a_k$.

1. Show that if $\{a_k\}_{k=1}^\infty$ converges to a, then $\{b_n\}_{n=1}^\infty$ converges to a.

2. Is the converse statement "if the Cesàro mean $\{b_n\}_{n=1}^\infty$ of a sequence $\{a_k\}_{k=1}^\infty$ converges, then the sequence $\{a_k\}_{k=1}^\infty$ also converges" true?

Proof. 1. Let $\varepsilon > 0$ be given. Since $\lim\limits_{k\to\infty} a_k = a$, there exists $N_1 > 0$ such that

$$\|a_k - a\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \geqslant N_1 .$$

Since $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{N_1} |a_k - a| = 0$, there exists $N_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1} \|a_k - a\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N_2 .$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,

$$\begin{aligned}\|b_n - a\| &= \Big\|\frac{1}{n}\sum_{k=1}^n a_k - a\Big\| \leqslant \frac{1}{n}\sum_{k=1}^n \|a_k - a\| \leqslant \frac{1}{n}\sum_{k=1}^{N_1}\|a_k - a\| + \frac{1}{n}\sum_{k=N_1}^n \|a_k - a\| \\ &< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1}^n \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{n - N_1 + 1}{n} < \varepsilon .\end{aligned}$$

2. No. For example, let $a_n = (-1)^n$. Then $\lim\limits_{n\to\infty} b_n = 0$ but $\{a_n\}_{n=1}^\infty$ does not converge. □

Problem 4. Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$; that is, every vector $\mathbf{e} \in \mathcal{V}$ can be expressed (uniquely) by $\boldsymbol{x} = \sum\limits_{i=1}^n x_i\mathbf{e}_i$ with $x_i \in \mathbb{F}$ for all $1 \leqslant i \leqslant n$. From Example 2.28 in the lecture note we know that $\langle\cdot,\cdot\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{F}$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{j=1}^n x_i\overline{y_i} \qquad \text{if } \boldsymbol{x} = \sum_{j=1}^n x_j\mathbf{e}_j \text{ and } \boldsymbol{y} = \sum_{j=1}^n y_j\mathbf{e}_j.$$

is an inner product. Show that $(\mathcal{V}, \|\cdot\|_2)$ is a Banach space, where $\|\cdot\|_2$ is the norm induced by the inner product.

Proof. Let $\{\boldsymbol{x}_k\}_{k=1}^\infty$ be a Cauchy sequence in $(\mathcal{V}, \|\cdot\|_2)$. Write $\boldsymbol{x}_k = x_k^{(1)}\mathbf{e}_1 + x_k^{(2)}\mathbf{e}_2 + \cdots + x_k^{(n)}\mathbf{e}_n$. For each $1 \leqslant j \leqslant n$ and $k, \ell \in \mathbb{N}$, The Cauchy-Schwarz inequality implies that

$$\big|x_k^{(j)} - x_\ell^{(j)}\big| = \big|\langle \boldsymbol{x}_k - \boldsymbol{x}_\ell, \mathbf{e}_j\rangle\big| \leqslant \|\boldsymbol{x}_k - \boldsymbol{x}_\ell\|_2\|\mathbf{e}_j\|_2 = \|\boldsymbol{x}_k - \boldsymbol{x}_\ell\|_2 ;$$

thus $\big\{x_k^{(j)}\big\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$ for all $1 \leqslant j \leqslant n$. Since $\mathbb{F}$ is complete, there exists $x_j \in \mathbb{F}$ such that

$$\lim_{k\to\infty} x_k^{(j)} = x^{(j)} .$$

Let $\varepsilon > 0$ be given. Since $\lim_{k\to\infty} x_k^{(j)} = x^{(j)}$ for $1 \leqslant j \leqslant n$, for each $1 \leqslant j \leqslant n$ there exists $N_j > 0$ such that

$$\left|x_k^{(j)} - x^{(j)}\right| < \frac{\varepsilon}{\sqrt{n}} \quad \text{whenever} \quad k \geqslant N_j\,.$$

Let $\boldsymbol{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Then for $k \geqslant N \equiv \max\{N_1, N_2, \cdots, N_n\}$,

$$\|\boldsymbol{x}_k - \boldsymbol{x}\|_2 \leqslant \sqrt{(x_k^{(1)} - x_1)^2 + \cdots + (x_k^{(n)} - x_n)^2} < \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon$$

which shows that $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ converges to $\boldsymbol{x}$. □

Problem 5. Show that $(\mathbb{C}, |\cdot|)$ is complete; that is, every Cauchy sequence $\{z_k\}_{k=1}^{n}$ in $\mathbb{C}$ converges.

Proof. Let $\{z_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{C}$. Write $z_n = x_n + iy_n$, where x_n and y_n are real numbers. Then

$$|x_n - x_m| \leqslant |z_n - z_m| \quad \text{and} \quad |y_n - y_m| \leqslant |z_n - z_m| \qquad \forall\, n, m \in \mathbb{N}\,.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}$; thus by the completeness of $\mathbb{R}$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ for some $x, y \in \mathbb{R}$. Let $z = x + yi$. Then

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0 \qquad \text{as} \quad n \to \infty\,;$$

thus we establish that every Cauchy sequence in $\mathbb{C}$ converges to a point in $\mathbb{C}$. This implies that $(\mathbb{C}, |\cdot|)$ is complete. □

Problem 6. Let (M, d) be a metric space. Two Cauchy sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ in M are said to be equivalent, denoted by $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$, if

$$\lim_{n\to\infty} d(p_n, q_n) = 0\,.$$

1. Prove that $\sim$ is an equivalence relation; that is, show that

   (a) $\{p_n\}_{n=1}^{\infty} \sim \{p_n\}_{n=1}^{\infty}$.

   (b) If $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$, then $\{q_n\}_{n=1}^{\infty} \sim \{p_n\}_{n=1}^{\infty}$.

   (c) If $\{p_n\}_{n=1}^{\infty} \sim \{q_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty} \sim \{r_n\}_{n=1}^{\infty}$, then $\{p_n\}_{n=1}^{\infty} \sim \{r_n\}_{n=1}^{\infty}$.

2. Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Show that the sequence $\big\{d(p_n, q_n)\big\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$; thus is convergent.

3. Let M^* be the set of all equivalence classes. If $P, Q \in M^*$, we define

$$d^*(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,,$$

   where $\{p_n\}_{n=1}^{\infty} \in P$ and $\{q_n\}_{n=1}^{\infty} \in Q$. Show that the definition above is well-defined; that is, show that if $\{p'_n\}_{n=1}^{\infty} \in P$ and $\{q'_n\}_{n=1}^{\infty} \in Q$ are another two Cauchy sequences, then $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$.

4. Define $\varphi : M \to M^*$ as follows: for each $x \in M$, $\{x_n\}_{n=1}^{\infty}$, where $x_n \equiv x$ for all $n \in \mathbb{N}$, is a Cauchy sequence in M. Then $\{x_n\}_{n=1}^{\infty} \in \varphi(x)$ for one particular $\varphi(x) \in M^*$. In other words, $\varphi(x)$ is the equivalence class where $\{x_n\}_{n=1}^{\infty}$ belongs to. Show that

$$d^*\big(\varphi(x), \varphi(y)\big) = d(x, y) \qquad \forall\, x, y \in M.$$

5. Show that $\varphi(M)$ is dense in M^*.

6. Show that (M^*, d^*) is a complete metric space. The metric space (M^*, d^*) is called the completion of (M, d).