Exercise Problem Sets 3

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Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field (satisfying the least upper bound property), and A be a non-empty subset of $\mathbb{F}$. Show that if $a \in A$ is an upper bound for A, then a is the least upper bound for A.

Proof. Let $\varepsilon > 0$ be given. There exists $x \in A$, namely $x = a$, such that $x > a - \varepsilon$. In other words, $a - \varepsilon$ cannot be an upper bound for A for all $\varepsilon > 0$; thus the fact that a is an upper bound for A implies that a is the least upper bound for A. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and A be a non-empty set of $\mathbb{F}$ which is bounded below. Define the set $-A$ by $-A \equiv \big\{-x \in \mathbb{F} \,\big|\, x \in A\big\}$. Prove that

$$\inf A = -\sup(-A)\,.$$

Note that this problem shows that if $\mathbb{F}$ satisfies the least upper bound property, then $\mathbb{F}$ satisfies the greatest lower bound property.

Proof. Let C be a subset of $\mathbb{F}$. Then

$$\begin{aligned} &b \text{ is a lower bound for a set } C \Leftrightarrow b \leqslant c \text{ for all } c \in C \Leftrightarrow -b \geqslant -c \text{ for all } c \in C \\ &\quad \Leftrightarrow -b \geqslant -c \text{ for all } -c \in -C \Leftrightarrow -b \geqslant c \text{ for all } c \in -C \Leftrightarrow -b \text{ is an upper bound for } -C. \end{aligned}$$

Therefore, we conclude that

$$b \text{ is a lower bound for a set } C \text{ if and only if } -b \text{ is an upper bound for } -C. \qquad (\star)$$

Now, since A is bounded from below implies that $-A$ is bounded from above. The least upper bound property then implies that $b = \sup(-A) \in \mathbb{F}$ exists. From $(\star)$, we find that $-b$ is a lower bound for A. Suppose that $-b$ is not the greatest lower bound for A. Then there exists $m > -b$ such that $m \leqslant x$ for all $x \in A$. This implies that m is a lower bound for A; thus $(\star)$ shows that $-m$ is an upper bound for $-A$. By the fact that $-m < b$, we conclude that b is not the least upper bound for $-A$, a contradiction to that b is the least upper bound for $-A$. □

Problem 3. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field. A number $x \in \mathbb{F}$ is called an ***accumulation point*** of a set $A \subseteq \mathbb{F}$ if for all $\delta > 0$, $(x - \delta, x + \delta)$ contains at least one point of A distinct from x. In logic notation,

$$x \text{ is an accumulation point of } A \quad \Leftrightarrow \quad (\forall\, \delta > 0)\big(A \cap (x-\delta, x+\delta)\backslash\{x\} \neq \varnothing\big)\,.$$

1. Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{F}$ so that $x_i \neq x_j$ for all $i, j \in \mathbb{N}$ and $A = \big\{x_k \,\big|\, k \in \mathbb{N}\big\}$, then x is an accumulation of A if and only if x is a cluster point of $\{x_n\}_{n=1}^{\infty}$.

2. How about if the condition $x_i \neq x_j$ for all $i, j \in \mathbb{N}$ is removed? Is the statement in 1 still valid?

Proof. 1. We show that

$$x \text{ is an accumulation point of } A \text{ if and only if } (\forall\, \delta > 0)\big(\#(A \cap (x-\delta, x+\delta)) = \infty\big)\,.$$

The direction "$\Leftarrow$" is trivial since if $\#(A \cap (x-\delta, x+\delta)) = \infty$, $A \cap (x-\delta, x+\delta)$ contains some point distinct from x.

($\Rightarrow$) Let $\delta_1 = 1$, by the definition of the accumulation points, there exists $x_1 \in A \cap (x-\delta_1, x+\delta_1)$ and $x_1 \neq x$. Define $\delta_2 = \min\big\{|x_1 - x|, \frac{1}{2}\big\}$. Then $\delta_2 > 0$; thus there exists $x_2 \in A \cap (x-\delta_2, x+\delta_2)$ and $x_2 \neq x$. We continue this process and obtain a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{x\}$ satisfying that

$$x_1 \in A \cap (x-1, x+1), \quad x_n \in A \cap (x-\delta_n, x+\delta_n) \text{ with } \delta_n = \min\big\{|x - x_{n-1}|, \frac{1}{n}\big\}\,.$$

By Archimedean property, $\{x_n\}_{n=1}^{\infty}$ converges to x since $|x - x_n| < \delta_n \leqslant \frac{1}{n}$. Let $\delta > 0$ be given. There exists $N > 0$ such that $\frac{1}{N} < \delta$; thus

$$A \cap (x-\delta, x+\delta) \supseteq A \cap \big(x - \frac{1}{N}, x + \frac{1}{N}\big) \supseteq \{x_N, x_{N+1}, x_{N+2}, \cdots\}\,.$$

Since $x_i \neq x_j$ for all $i, j \in \mathbb{N}$, we must have $\#\big(A \cap (x-\delta, x+\delta)\big) = \infty$. □

Problem 4. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{F}$. Show that the following three statements are equivalent.

1. $\{x_n\}_{n=1}^{\infty}$ converges.

2. Every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges.

3. Every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges.

Proof. "1 $\Rightarrow$ 2" Suppose that $\{x_n\}_{n=1}^{\infty}$ converges. Then $\{x_n\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{F}$; thus Proposition 1.60 of the lecture note shows that every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x. Therefore, every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges.

"2 $\Rightarrow$ 3" Suppose that every proper subsequence of $\{x_n\}_{n=1}^{\infty}$ converges. Then $\{x_{n+1}\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{F}$; thus $\{x_n\}_{n=1}^{\infty}$ converges to x. This implies that every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges (since $\{x_n\}_{n=1}^{\infty}$ is the only non-proper subsequence of $\{x_n\}_{n=1}^{\infty}$).

"3 $\Rightarrow$ 1" Suppose that every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges. In particular, the fact that $\{x_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ implies that $\{x_n\}_{n=1}^{\infty}$ converges. □

Problem 5. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ be a sequence satisfying $|x_n - x_{n+1}| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove or disprove that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ so that $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence.

Solution. Let $\mathbb{F} = \mathbb{R}$, and define sequence $\{x_n\}_{n=1}^\infty$ as follows: $x_1 = 0$ and for each $n \in \mathbb{N}$,

$$x_{n+1} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k}\,.$$

Then $|x_1 - x_2| = \frac{1}{2} < \frac{1}{1}$ and $|x_n - x_{n+1}| = \frac{1}{2} \cdot \frac{1}{n} < \frac{1}{n}$ for all $n \geqslant 2$. Therefore, the sequence $\{x_n\}_{n=1}^\infty$ satisfies the required properties. However, such an $\{x_n\}_{n=1}^\infty$ is an increasing sequence which is not bounded from above so that any subsequence of $\{x_n\}_{n=1}^\infty$ is also increasing and is not bounded from above. Therefore, any subsequence of $\{x_n\}_{n=1}^\infty$ diverges; thus any subsequence of $\{x_n\}_{n=1}^\infty$ cannot be Cauchy sequence (since Cauchy sequence in $\mathbb{R}$ must converge). □

Problem 6. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $f : \mathbb{F} \to \mathbb{F}$ be a function so that

$$|f(x) - f(y)| \leqslant \alpha |x - y| \qquad \forall\, x, y \in \mathbb{F}\,,$$

where $\alpha \in \mathbb{F}$ is a constant satisfying $0 < \alpha < 1$. Pick an arbitrary $x_1 \in \mathbb{F}$, and define $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$.

Proof. Since $0 < \alpha < 1$, Problem 2 in Exercise 2 shows that $\lim\limits_{n\to\infty} \alpha^n = 0$. By the fact that $|f(x) - f(y)| \leqslant \alpha|x - y|$ and $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leqslant \alpha |x_n - x_{n-1}| \qquad \forall\, n \geqslant 2\,;$$

thus

$$|x_{n+1} - x_n| \leqslant \alpha |x_n - x_{n-1}| \overset{(\text{if } n \geqslant 3)}{\leqslant} \alpha^2 |x_{n-1} - x_{n-2}| \leqslant \cdots \leqslant \alpha^{n-1} |x_2 - x_1|\,.$$

Therefore, if $n > m$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leqslant \alpha^{n-2}|x_2 - x_1| + \alpha^{n-3}|x_2 - x_1| + \cdots + \alpha^{m-1}|x_2 - x_1| \\
&= \big(\alpha^{n-2} + \alpha^{n-3} + \alpha^{m-1}\big)|x_2 - x_1| \leqslant \frac{\alpha^{m-1}}{1 - \alpha}|x_2 - x_1|\,.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \alpha^n = 0$, there exists $N > 0$ such that

$$\frac{\alpha^{n-1}}{1 - \alpha}|x_2 - x_1| < \varepsilon \quad \text{ whenever } \quad n \geqslant N\,.$$

Then if $n > m \geqslant N$, by the fact that $|x_n - x_m| \leqslant \frac{\alpha^{m-1}}{1 - \alpha}|x_2 - x_1|$ we obtain that $|x_n - x_m| < \varepsilon$. □

Problem 7. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field with Archimedean Property, $I \subseteq \mathbb{F}$ be a non-empty interval, and $f : I \to \mathbb{F}$ be a function.

1. f is said to have a limit at $c \in I$ or we say that the limit of f at c exists if

   $$\lim_{n\to\infty} f(x_n) \text{ exists for all convergent sequences } \{x_n\}_{n=1}^\infty \subseteq I \backslash \{c\} \text{ with limit } c.$$

   Show that the limit of f at c exists if and only if there exists $L \in \mathbb{F}$ satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$\big|f(x) - L\big| < \varepsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

2. f is said to be continuous at a point $c \in I$ if

$$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

Show that f is continuous at c if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - f(c)\big| < \varepsilon \qquad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\,.$$

Proof. 1. ("$\Rightarrow$") Suppose that the limit of f at c exists.

Claim: If $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = c$, then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(y_n)$.

Proof of claim: Define z_n by

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\,, \\ y_{\frac{n}{2}} & \text{if } n \text{ is even}\,. \end{cases}$$

Then $\lim\limits_{n\to\infty} z_n = c$; thus by the assumption that the limit of f at c exists, we find that $\lim\limits_{n\to\infty} f(z_n)$ exists. On the other hand, since $\lim\limits_{n\to\infty} f(x_n)$ and $\lim\limits_{n\to\infty} f(y_n)$ both exist, we must have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(y_n)\,. \qquad \square$$

Having established the claim, we find that there exists $L \in \mathbb{F}$ such that $\lim\limits_{n\to\infty} f(x_n) = L$ whenever $\{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ is a convergent sequence with limit c.

Suppose the contrary that there exists $\varepsilon > 0$ such that for each $\delta > 0$ there exists $x \in I$ satisfying $0 < |x - c| < \delta$ and $\big|f(x) - L\big| \geqslant \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in I$ satisfying

$$0 < |x_n - c| < \frac{1}{n} \qquad \text{and} \qquad \big|f(x_n) - L\big| \geqslant \varepsilon\,.$$

Then $\{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ and Archimedean Property implies that $\lim\limits_{n\to\infty} x_n = c$. Therefore, the claim shows that $\lim\limits_{n\to\infty} f(x_n) = L$ which contradicts to the inequality $\big|f(x_n) - L\big| \geqslant \varepsilon$.

("$\Leftarrow$") Let $\{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ be a convergent sequence with limit c, and $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

By the fact that $\lim\limits_{n\to\infty} x_n = c$, there exists $N > 0$ such that

$$|x_n - c| < \delta \quad \text{whenever} \quad n \geqslant N\,.$$

Therefore, if $n \geqslant N$, then $0 < |x_n - c| < \delta$ and $x_n \in I$ so that $\big|f(x_n) - L\big| < \varepsilon$. This implies that $\lim\limits_{n\to\infty} f(x_n) = L$; thus

$$\lim_{n\to\infty} f(x_n) \text{ exists for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\} \text{ with limit } c.$$

2. ("$\Rightarrow$") Suppose that f is continuous at a point $c \in I$; that is,

$$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

In particular, for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\}$ with limit c we have $\lim\limits_{n\to\infty} f(x_n) = f(c)$. Therefore, 1 implies that

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\Big)\,.$$

We note that we must have $\big|f(c) - f(c)\big| < \varepsilon$; thus the statement above implies that

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\Big)\,.$$

("$\Leftarrow$") We note that the assumption in particular implies that

$$(\forall\,\varepsilon > 0)(\exists\,\delta > 0)\Big(\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\Big)\,;$$

thus 1 implies that

$$\lim_{n\to\infty} f(x_n) = f(c) \text{ for all convergent sequences } \{x_n\}_{n=1}^{\infty} \subseteq I\backslash\{c\} \text{ with limit } c. \tag{0.1}$$

Now suppose the contrary that there exists a convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq I$ with limit c but $\lim\limits_{n\to\infty} f(x_n) \neq f(c)$. Then (0.1) implies that

$$\#\{n \in \mathbb{N} \,|\, x_n = c\} = \infty\,.$$

(a) If $\#\{n \in \mathbb{N} \,|\, x_n \neq c\} < \infty$, then there exists $N > 0$ such that $x_n = c$ for all $n \geqslant N$. This implies that $\big|f(x_n) - f(c)\big| = 0 < \varepsilon$ whenever $n \geqslant$N, a contradiction to that $\lim\limits_{n\to\infty} f(x_n) \neq f(c)$.

(b) If $\#\{n \in \mathbb{N} \,|\, x_n \neq c\} = \infty$, then $\{n \in \mathbb{N} \,|\, x_n \neq c\} = \{n_j \in \mathbb{N} \,|\, j \in \mathbb{N}, n_j < n_{j+1}\}$ and $\{x_{n_j}\}_{j=1}^{\infty} \subseteq I\backslash\{c\}$ is a convergent sequence with limit c. Therefore, (0.1) implies that

$$\lim_{j\to\infty} f\big(x_{n_j}\big) = f(c)\,.$$

Let $\varepsilon > 0$ be given. The limit above shows that there exists $J > 0$ such that $\big|f(x_{n_j}) - f(c)\big| < \varepsilon$ whenever $j \geqslant J$. Let $N = n_J$. Then for all $n \geqslant N$, we have either $x_n = c$ or $x_n = x_{n_j}$ for some $j \geqslant J$; thus

$$\big|f(x_n) - f(c)\big| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,,$$

a contradiction to that $\lim\limits_{n\to\infty} f(x_n) \neq f(c)$. □