Exercise Problem Sets 2

Sept. 23. 2022

Problem 1. Complete the following.

1. Verify the Wallis's formula: if n is a non-negative integer, then
$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \frac{(2^n n!)^2}{(2n+1)!}$$
and
$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.$$

2. Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Show that $\displaystyle\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1$.

3. Let $s_n = \dfrac{n!}{n^{n+0.5}e^{-n}}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence; that is, $s_n \geqslant s_{n+1}$ for all $n \in \mathbb{N}$.

4. Suppose that you know that $\mathbb{R}$ satisfies **MSP**. Then explain why the limit $\displaystyle\lim_{n\to\infty} s_n$ exists. Find the limit of $\{s_n\}_{n=1}^{\infty}$.

Hint:

2. Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ for all $n \in \mathbb{N}$ and then apply the Sandwich lemma.

3. Consider the function $f(x) = \left(1+\dfrac{1}{x}\right)^{x+0.5}$.

Proof. 1. Integrating by parts, we find that
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x\,dx &= -\sin^{n-1} x\cos x\Big|_{x=0}^{x=\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x\,dx\\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)\,dx\\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx\,;\end{aligned}$$
thus
$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx\,.$$
Therefore,
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx &= \frac{2n}{2n+1}\int_0^{\frac{\pi}{2}} \sin^{2n-1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\frac{\pi}{2}} \sin^{2n-3} x\,dx = \cdots\\ &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1}\\ &= \frac{2^2 4^2\cdots(2n)^2}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx &= \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \sin^{2n-2} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\frac{\pi}{2}} \sin^{2n-4} x\,dx = \cdots \\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{0} x\,dx = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2} \\ &= \frac{(2n)!}{2^2 4^2\cdots(2n)^2}\cdot\frac{\pi}{2} = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.\end{aligned}$$

2. On the interval $\big[0,\frac{\pi}{2}\big]$, $0 \leqslant \sin x \leqslant 1$; thus

$$\sin^{2n+2} x \leqslant \sin^{2n+1} x \leqslant \sin^{2n} x \qquad \forall\, x \in \big[0,\frac{\pi}{2}\big]\,.$$

Therefore, $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ so that

$$\frac{I_{2n+2}}{I_{2n}} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1 \qquad \forall\, n \in \mathbb{N}\,.$$

Since $\dfrac{I_{2n+2}}{I_{2n}} = \dfrac{2n+1}{2(n+1)}$, the Sandwich Lemma implies that

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1\,.$$

3. Since $\lim\limits_{n\to\infty}\Big(1+\dfrac{1}{n}\Big)^{n+0.5} = e$ and $\dfrac{s_n}{s_{n+1}} = \dfrac{\dfrac{n!}{n^{n+0.5}e^{-n}}}{\dfrac{(n+1)!}{(n+1)^{n+1.5}e^{-n-1}}} = \dfrac{1}{e}\Big(1+\dfrac{1}{n}\Big)^{n+0.5}$, it suffices to show that the function $f(x) \equiv \big(1+\dfrac{1}{x}\big)^{x+0.5}$ is decreasing on $[1,\infty)$. Nevertheless, this is the same as proving that the function $g(x) \equiv (1+x)^{\frac{1}{x}+\frac{1}{2}}$ is increasing on $(0,1]$.

Differentiate g, we find that

$$\begin{aligned} g'(x) &= g(x)\frac{\big[\ln(1+x)+\frac{2+x}{1+x}\big]2x - 2(2+x)\ln(1+x)}{4x^2} \\ &= \frac{2x+x^2-2(1+x)\ln(1+x)}{2x^2(1+x)}\,.\end{aligned}$$

To see the sign of the denominator $h(x) = 2x+x^2-2(1+x)\ln(1+x)$ on $(0,1]$, we differentiate h and find that

$$h'(x) = 2+2x-2\ln(1+x)-2 = 2\big[x-\ln(1+x)\big]$$

and one more differentiation shows that

$$h''(x) = 1-\frac{1}{1+x} = \frac{x}{1+x} > 0 \qquad \forall\, x \in (0,1]\,.$$

Therefore, h' in increasing on $(0,1]$ which implies that $h'(x) \geqslant h'(0) = 0$ for all $x \in (0,1]$. This further implies that $h(x) \geqslant h(0) = 0$ for all $x \in (0,1]$; thus $g'(x) \geqslant 0$ for all $x \in \big(0,1\big]$.

4. Since $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence and is bounded from below. By the monotone sequence property, $\lim\limits_{n\to\infty} s_n = s$ exists. Note that

$$\begin{aligned}\frac{I_{2n+1}}{I_{2n}} &= \frac{2}{\pi}\frac{(2^n n!)^4}{(2n)!(2n+1)!} = \frac{2^{4n+1}}{\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(n^{n+0.5}e^{-n})^4}{(2n)^{2n+0.5}e^{-2n}(2n+1)^{2n+1.5}e^{-(2n+1)}}\\ &= \frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(2n)^{2n+1.5}}{(2n+1)^{2n+1.5}} = \frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\Big(1+\frac{1}{2n}\Big)^{-2n-1.5}.\end{aligned}$$

Therefore, 2 implies that

$$1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(2n)^{2n+1.5}}{(2n+1)^{2n+1.5}} = \frac{e}{2\pi}s^2\lim_{n\to\infty}\Big(1+\frac{1}{2n}\Big)^{-2n-1.5} = \frac{s^2}{2\pi};$$

thus $s = \sqrt{2\pi}$ (since $s_n \geqslant 0$). □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $0 < \alpha < 1$. Show that $\lim\limits_{n\to\infty} \alpha^n = 0$.

Proof. Since $0 < \alpha < 1$, we have $\frac{1}{\alpha} > 1$; thus by the fact that $\lim\limits_{n\to\infty}\frac{1}{n} = 0$ (which is from Archimedean property), there exists $p > 0$ such that

$$1 + \frac{1}{p} < \frac{1}{\alpha}.$$

Therefore,

$$\frac{1}{\alpha^p} > \Big(1+\frac{1}{p}\Big)^p \geqslant 1 + C_1^p\frac{1}{p} = 2$$

which implies that

$$0 < \alpha^p < \frac{1}{2}.$$

By the fact that $2^n \geqslant n$ for all $n \geqslant \mathbb{N}$ (which can be shown by induction), we find from the Sandwich Lemma that

$$\lim_{n\to\infty}\alpha^{pn} = 0.$$

Let $\varepsilon > 0$ be given. The identity above shows the existence of $N_1 > 0$ such that $|\alpha^{pn}| < \varepsilon$ whenever $n \geqslant N_1$. Let $N = pN_1$. Then if $n \geqslant N$,

$$|\alpha^n| \leqslant |\alpha^{pN_1}| < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty}\alpha^n = 0$. □

Problem 3. Let $(\mathbb{F}, +, \cdot, \leqslant)$ an ordered field satisfying the monotone sequence property, and $y \in \mathbb{F}$ satisfying $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in the last example in class).

2. Show that $y^n - 1 > n(y-1)$ for all $n \in \mathbb{N}\backslash\{1\}$; thus $y - 1 > n(y^{1/n} - 1)$.

3. Show that if $t > 1$ and $n > (y-1)/(t-1)$, then $y^{1/n} < t$.

4. Show that $\lim\limits_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

Proof. 1. For each $k \in \mathbb{N}$, let N_k be the largest integer satisfying that $\big(\frac{N_k}{2^k}\big)^n \leqslant y$ but $\big(\frac{N_k+1}{2^k}\big)^n > y$ (the existence of such an N_k requires the Archimedean property, why?) Define $x_k = \frac{N_k}{2^k}$. Then

(a) By binomial expansion, for each $k \in \mathbb{N}$ we have

$$x_k^n \leqslant y < 1 + C_1^n y + C_2^n y^2 + \cdots + C_n^n y^n = (1+y)^n\,;$$

thus Problem 2 in Exercise 1 implies that $x_k < 1+y$. Therefore, $\{x_k\}_{k=1}^{\infty}$ is bounded from above.

(b) For each $k \in \mathbb{N}$, $\big(\frac{2N_k}{2^{k+1}}\big)^n = \big(\frac{N_k}{2^k}\big)^n \leqslant y$; thus $N_{k+1} \geqslant 2N_k$. Therefore, for each $k \in \mathbb{N}$,

$$x_k = \frac{N_k}{2^k} = \frac{2N_k}{2^{k+1}} \leqslant \frac{N_{k+1}}{2^{k+1}} = x_{k+1}$$

which shows that $\{x_k\}_{k=1}^{\infty}$ is increasing.

Therefore, **MSP** implies that $\{x_k\}_{k=1}^{\infty}$ converges. Assume that $x_k \to x$ as $k \to \infty$ for some $x \in \mathbb{F}$. Then the fact that $x_k^n \leqslant y$ for all $k \in \mathbb{N}$ implies that $x^n \leqslant y$. On the other hand,

$$\Big(x_k + \frac{1}{2^k}\Big)^n \geqslant y \qquad \forall\, k \in \mathbb{N}\,;$$

thus **AP** (a consequence of **MSP**) implies that

$$x^n = \Big(\lim_{k\to\infty} x_k + \lim_{k\to\infty}\frac{1}{2^k}\Big)^n = \lim_{k\to\infty}\Big(x_k + \frac{1}{2^k}\Big)^n \geqslant y\,.$$

Therefore, $x^n = y$. Problem 2 then shows that there is only one $x > 0$ satisfying $x^n = y$. This x will be denoted by $y^{\frac{1}{n}}$.

2. For $y > 1$, let $z = y - 1$. Then $z > 0$ so that for $n > 1$, the binomial expansion shows that

$$\begin{aligned} y^n - 1 &= (1+z)^n - 1 = 1 + C_1^n z + C_2^n z^2 + \cdots + C_n^n z^n - 1 = C_1^n z + C_2^n z^2 + \cdots + C_n^n z^n \\ &> nz = n(y-1)\,. \end{aligned}$$

Therefore, replacing y by $y^{\frac{1}{n}}$ in the inequality above, we conclude that

$$y - 1 > n(y^{\frac{1}{n}} - 1) \qquad \forall\, n \in \mathbb{N}\backslash\{1\}\,.$$

3. Suppose that $y^{\frac{1}{n}} \geqslant t > 1$. Then 2 implies that for $n \in \mathbb{N}\backslash\{1\}$,

$$y - 1 > n(y^{\frac{1}{n}} - 1) \geqslant n(t-1)\,.$$

Therefore, $n \leqslant \dfrac{y-1}{t-1}$, a contradiction.

4. Let $k \in \mathbb{N}$ and $t = 1 + \dfrac{1}{k}$ in 3. Then for $n > k(y-1)$,

$$1 \leqslant y^{\frac{1}{n}} < 1 + \frac{1}{k}\,.$$

Since $n \to \infty$ as $k \to \infty$, by the Sandwich Lemma we conclude that $\lim\limits_{n\to\infty} y^{\frac{1}{n}} = 1$. □

Problem 4. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and $S \subseteq \mathbb{F}$ be non-empty.

1. Show that if S is bounded from below, then

$$\inf S = \sup \big\{ x \in \mathbb{F} \,\big|\, x \text{ is a lower bound for } S \big\}$$

2. Show that if S is bounded from above, then

$$\sup S = \inf \big\{ x \in \mathbb{F} \,\big|\, x \text{ is an upper bound for } S \big\} .$$

Proof. Define $A = \big\{ x \in \mathbb{F} \,\big|\, x \text{ is a lower bound for } S \big\}$. Since S is non-empty, every element in S is an upper bound for A; thus A is bounded from above. By the least upper bound property, $b = \sup A \in \mathbb{F}$ exists. Note that by the definition of A,

$$\text{if } x \in A, \text{ then } x \leqslant s \text{ for all } s \in S. \tag{$\star$}$$

Let $\varepsilon > 0$ be given. Then $b - \varepsilon$ is not an upper bound for A; thus there exists $x \in A$ such that $b - \varepsilon < x$. Then ($\star$) implies that $b - \varepsilon < s$ for all $s \in S$. Since $\varepsilon > 0$ is given arbitrarily, $b \leqslant s$ for all $s \in S$; thus b is a lower bound for S.

Suppose that b is not the greatest lower bound for S. There exists $m > b$ such that $m \leqslant s$ for all $s \in S$. Therefore, $m \in A$; thus $m \leqslant b$, a contradiction. □

Problem 5. Let A, B be two sets, and $f : A \times B \to \mathbb{F}$ be a function, where $(\mathbb{F}, +, \cdot, \leqslant)$ is an ordered field satisfying the least upper bound property. Show that

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B} \big(\sup_{x\in A} f(x,y) \big) = \sup_{x\in A} \big(\sup_{y\in B} f(x,y) \big) .$$

Proof. Note that

$$f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, (x,y) \in A \times B ;$$

thus

$$\sup_{x\in A} f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, y \in B .$$

The inequality above further shows that

$$\sup_{y\in B} \Big(\sup_{x\in A} f(x,y) \Big) \leqslant \sup_{(x,y)\in A\times B} f(x,y) . \tag{$\star$}$$

Now we show the reverse inequality.

1. Suppose that $\sup\limits_{(x,y)\in A\times B} f(x,y) = M < \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that

$$f(x_k, y_k) > M - \frac{1}{k} .$$

Therefore,

$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{x \in A} f(x, y_k)$$

which further implies that

$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{y \in B} \big(\sup_{x \in A} f(x, y) \big) .$$

Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\sup\limits_{y \in B} \big(\sup\limits_{x \in A} f(x, y) \big) \geqslant M$.

2. Suppose that $\sup\limits_{(x,y) \in A \times B} f(x, y) = \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that

$$f(x_k, y_k) > k .$$

Therefore,

$$k < f(x_k, y_k) \leqslant \sup_{x \in A} f(x, y_k)$$

which further implies that

$$k < f(x_k, y_k) \leqslant \sup_{y \in B} \big(\sup_{x \in A} f(x, y) \big) .$$

Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\sup\limits_{y \in B} \big(\sup\limits_{x \in A} f(x, y) \big) = \infty$.

With the help of ($\star$), we conclude that $\sup\limits_{(x,y) \in A \times B} f(x, y) = \sup\limits_{y \in B} \big(\sup\limits_{x \in A} f(x, y) \big)$. □

Problem 6. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$. Define

$$\|\boldsymbol{x}\|_1 = \sum_{k=1}^{n} |x_k| \qquad \text{and} \qquad \|\boldsymbol{x}\|_\infty = \max \big\{ |x_1|, |x_2|, \cdots, |x_n| \big\} .$$

Show that

1. $\|\boldsymbol{x}\|_1 = \sup \Big\{ \sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1 \Big\}$.
2. $\|\boldsymbol{y}\|_\infty = \sup \Big\{ \sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1 \Big\}$.

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ be given. Then

$$\sum_{k=1}^{n} x_k y_k \leqslant \sum_{k=1}^{n} |x_k||y_k| \leqslant \sum_{k=1}^{n} |x_k| \|\boldsymbol{y}\|_\infty = \|\boldsymbol{y}\|_\infty \sum_{k=1}^{n} |x_k| = \|\boldsymbol{y}\|_\infty \|\boldsymbol{x}\|_1 .$$

Therefore,

$$\sup \Big\{ \sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1 \Big\} \leqslant \|\boldsymbol{x}\|_1 \qquad \text{and} \qquad \sup \Big\{ \sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1 \Big\} \leqslant \|\boldsymbol{y}\|_\infty .$$

Next we show that the two inequalities are in fact equalities by showing that the right-hand side of the inequalities belongs to the sets (this is because if $b \in A$ is an upper bound for A, then b is the least upper bound for A).

1. $\sup\left\{ \sum_{k=1}^n x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\right\} = \|\boldsymbol{x}\|_1$: W.L.O.G. we can assume that $\boldsymbol{x} \neq \mathbf{0}$. For a given $\boldsymbol{x} \in \mathbb{F}^n$, define $y_k \in \mathbb{F}$ by

$$y_k = \begin{cases} \dfrac{\overline{x_k}}{|x_k|} & \text{if } x_k \neq 0\,, \\ 0 & \text{if } x_k = 0\,, \end{cases}$$

where $\overline{x_k}$ denotes the complex conjugate of x_k. Then $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ satisfies $\|\boldsymbol{y}\|_\infty = 1$ (since at least one component of $\boldsymbol{x}$ is non-zero), and

$$\sum_{k=1}^n x_k y_k = \sum_{k=1}^n |x_k| = \|\boldsymbol{x}\|_1\,.$$

2. $\sup\left\{ \sum_{k=1}^n x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\right\} = \|\boldsymbol{y}\|_\infty$: W.L.O.G. we can assume that $\boldsymbol{y} \neq \mathbf{0}$. Suppose that $\|\boldsymbol{y}\|_\infty = |y_m| \neq 0$ for some $1 \leqslant m \leqslant n$; that is, the maximum of the absolute value of components occurs at the m-th component. Define $x_j \in \mathbb{F}$ by

$$x_j = \begin{cases} \dfrac{\overline{y_m}}{|y_m|} & \text{if } j = m\,, \\ 0 & \text{if } j \neq m\,, \end{cases}$$

where $\overline{y_m}$ is the complex conjugate of y_m. Then $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ satisfies $\|\boldsymbol{x}\|_1 = 1$ (since only one component of $\boldsymbol{x}$ is non-zero), and

$$\sum_{k=1}^n x_k y_k = \frac{\overline{y_m}}{|y_m|} y_m = |y_m| = \|\boldsymbol{y}\|_\infty\,.$$

□

Problem 7. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and A, B be non-empty subsets of $\mathbb{F}$. Define $A + B = \{x + y \,|\, x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(A + B) = \inf A + \inf B$.
3. $\sup(A \cap B) \leqslant \min\{\sup A, \sup B\}$.
4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
5. $\sup(A \cup B) \geqslant \max\{\sup A, \sup B\}$.
6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Proof. 1. Let $a = \sup A$, $b = \sup B$, and $\varepsilon > 0$ be given. W.L.O.G. we can assume that $a, b \in \mathbb{F}$ for otherwise $a = \infty$ or $b = \infty$ so that $A + B$ is not bounded from above.

(a) Let $z \in A + B$. Then $z = x + y$ for some $x \in A$ and $y \in B$. By the fact that $x \leqslant a$ and $y \leqslant b$, we find that $z \leqslant a + b$. Therefore, $a + b$ is an upper bound for $A + B$.

(b) There exists $x \in A$ and $y \in B$ such that $x > a - \dfrac{\varepsilon}{2}$ and $y > b - \dfrac{\varepsilon}{2}$; thus there exists $z = x + y \in A + B$ such that

$$z = x + y > a + b - \varepsilon\,.$$

Therefore, $a + b = \sup(A + B)$.

2. By Problem 1,

$$\begin{aligned}\inf(A+B) &= -\sup(-(A+B)) = -\sup(-A+(-B)) = -\sup(-A) - \sup(-B) \\ &= \inf(A) + \inf(B)\,.\end{aligned}$$

3. The desired inequality hold if $A \cap B = \varnothing$ (since then $\sup A \cap B = -\infty$), so we assume that $A \cap B \neq \varnothing$. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Therefore,

$$\sup(A \cap B) \leqslant \sup A \qquad \text{and} \qquad \sup(A \cap B) \leqslant \sup B\,.$$

The inequalities above then implies that $\sup(A \cap B) \leqslant \min\{\sup A, \sup B\}$.

4. If A and B are non-empty bounded sets but $A \cap B = \varnothing$, then $\sup(A \cap B) = -\infty$ but $\sup A, \sup B \in \mathbb{F}$. In such a case $\sup(A \cap B) \neq \min\{\sup A, \sup B\}$.

5. Similar to 3, we have $A \subseteq A \cup B$ and $B \subseteq A \cup B$; thus

$$\sup A \leqslant \sup(A \cup B) \qquad \text{and} \qquad \sup B \leqslant \sup(A \cup B)\,.$$

Therefore, $\max\{\sup A, \sup B\} \leqslant \sup(A \cup B)$.

6. If one of A and B is not bounded from above, then $\sup(A \cup B) = \max\{\sup A, \sup B\} = \infty$. Suppose that A and B are bounded from above. Then $A \cup B$ are bounded from above by $\max\{\sup A, \sup B\}$ since if $x \in A \cup B$, then $x \in A$ or $x \in B$ which implies that $x \leqslant \sup A$ or $x \leqslant \sup B$; thus $x \leqslant \max\{\sup A, \sup B\}$ for all $x \in A \cup B$. This shows that

$$\sup(A \cup B) \leqslant \max\{\sup A, \sup B\}\,.$$

Together with 5, we conclude that $\sup(A \cup B) = \max\{\sup A, \sup B\}$. $\square$