Exercise Problem Sets 1

Sept. 16. 2022

Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b \in \mathbb{F}$. Show that $a \leqslant b$ if and only if for all $\varepsilon > 0$, $a < b + \varepsilon$.

Proof. "$\Rightarrow$" Let $\varepsilon > 0$ be given. By the compatibility of $\leqslant$ and $+$,

$$a < a + \varepsilon \leqslant b + \varepsilon$$

which implies that $a < b + \varepsilon$.

"$\Leftarrow$" Suppose the contrary that $a > b$. Let $\varepsilon = a - b$. Then $\varepsilon > 0$; thus

$$a < b + (a - b) = a\,,$$

a contradiction. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $x, y \in \mathbb{F}$, and $n \in \mathbb{N}$. Show that

1. If $0 \leqslant x < y$, then $x^n < y^n$.

2. If $0 \leqslant x, y$ and $x^n < y^n$, then $x < y$.

Proof. 1. Let $S = \{n \in \mathbb{N} \,|\, x^n < y^n\}$. Then $1 \in S$ by assumption. Suppose that $n \in S$. Then $0 \leqslant x^n < y^n$. By the fact that $0 \leqslant x < y$, we find that

$$x^{n+1} = x^n \cdot x < x^n \cdot y < y^n \cdot y = y^{n+1}\,;$$

thus $n + 1 \in S$. By induction, we conclude that $S = \mathbb{N}$.

2. Suppose the contrary that $x \geqslant y$. Then 1 implies that $x^n \geqslant y^n$, a contradiction. □

Problem 3. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the Archimedean property, and $x, y \in \mathbb{F}$ satisfying $x < y$. Show that there exists $r \in \mathbb{Q}$ such that $x < r < y$. This property is called the ***denseness*** of $\mathbb{Q}$ (in Archimedean ordered fields).

Proof. If $x < 0 < y$, then we can simply choose $r = 0$. It then suffices to establish the case for $0 < x < y$ (since if $x < y < 0$ we pick r satisfying $-y < r < -x$ so that $x < -r < y$).

Since $y - x > 0$, by the Archimedean property there exists $n \in \mathbb{N}$ such that $\frac{1}{y - x} < n$. This implies that $nx + 1 < ny$ for such n.

Let $S = \{m \in \mathbb{N} \,|\, nx < m\}$. By the Archimedean property, $S \neq \varnothing$; thus the well-ordering principle implies that $m = \min S$ exists. Such m satisfies

$$nx < m \leqslant nx + 1 < ny\,;$$

thus $x < \frac{m}{n} < y$. The number $r = \frac{m}{n}$ is one of the desired rational numbers. □

Alternative proof. If $x < 0 < y$, then we can simply choose $r = 0$. It then suffices to establish the case for $0 < x < y$ (since if $x < y < 0$ we pick r satisfying $-y < r < -x$ so that $x < -r < y$).

Suppose the contrary that there are $0 < x < y$ such that no rational number r satisfy $x < r < y$. Let $n \in \mathbb{N}$ be given. Define $S = \{k \in \mathbb{N} \,|\, nx < k\}$. The Archimedean property implies that $S \neq \varnothing$; thus $m = \min S$ exists. Since there is no rational number in between x and y, such m satisfies

$$\frac{m-1}{n} < x < y < \frac{m}{n};$$

thus $0 < y - x < \frac{1}{n}$. Therefore, we establish that $\frac{1}{y-x}$ is an upper bound for $\mathbb{N}$, a contradiction to the Archimedean property. □