Exercise Problem Sets 11

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Problem 1. Suppose that the Fourier transform of $f \in \mathscr{S}(\mathbb{R})$ is $\widehat{f}(\xi)$. Find the Fourier transform of the function $y = f(2x+1)\cos x$.

Proof. By the Euler identity, $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}} f(2x+1)\cos x e^{-ix\xi}\,dx &= \int_{\mathbb{R}} f(2x+1)\frac{e^{ix}+e^{-ix}}{2}e^{-ix\xi}\,dx \\
&= \frac{1}{2}\Big(\int_{\mathbb{R}} f(2x+1)e^{-ix(\xi-1)}\,dx + \int_{\mathbb{R}} f(2x+1)e^{-ix(\xi+1)}\,dx\Big) \\
&= \frac{1}{4}\Big(\int_{\mathbb{R}} f(t)e^{-i\frac{t-1}{2}(\xi-1)}\,dt + \int_{\mathbb{R}} f(t)e^{-i\frac{t-1}{2}(\xi+1)}\,dt\Big) \\
&= \frac{1}{4}\Big(e^{i\frac{\xi-1}{2}}\int_{\mathbb{R}} f(t)e^{-it\frac{\xi-1}{2}}\,dt + e^{i\frac{\xi+1}{2}}\int_{\mathbb{R}} f(t)e^{-it\frac{\xi+1}{2}}\,dt\Big)
\end{aligned}
$$

which shows that the Fourier of $y = f(2x+1)\cos x$ is $\dfrac{1}{4}\Big[e^{i\frac{\xi-1}{2}}\widehat{f}\big(\dfrac{\xi-1}{2}\big) + e^{i\frac{\xi+1}{2}}\widehat{f}\big(\dfrac{\xi+1}{2}\big)\Big]$. □

Problem 2. A vector-valued function $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$ is called a Schwartz function, still denoted by $\boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)$, if $u_j \in \mathscr{S}(\mathbb{R}^n)$ for all $1 \leqslant j \leqslant n$. Show the Korn inequality

$$
\sum_{i,j=1}^{n} \big\|\epsilon_{ij}(\boldsymbol{u})\big\|^2_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{2}\sum_{i,j=1}^{n}\Big\|\frac{\partial u_j}{\partial x_i}\Big\|^2_{L^2(\mathbb{R}^n)} \qquad \forall\, \boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)\,,
$$

where $\epsilon_{ij}(\boldsymbol{u}) = \dfrac{1}{2}\big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\big)$ is the symmetric part of $D\boldsymbol{u}$.

Hint: Use the Plancherel formula.

Proof. By the Plancherel formula,

$$
\begin{aligned}
\big\|\epsilon_{ij}(u)\big\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{4}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\big[\xi_i\xi_i\widehat{u}_j(\xi)\overline{\widehat{u}_j(\xi)} + \xi_j\xi_j\widehat{u}_i(\xi)\overline{\widehat{u}_i(\xi)} + \xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)} + +\xi_j\xi_i\overline{\widehat{u}_i(\xi)}\widehat{u}_j(\xi)\big]d\xi \\
&= \sum_{i=1}^{n}\int_{\mathbb{R}^n}\xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n}\big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 + 2\xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)}\big]d\xi \\
&\geqslant \sum_{i=1}^{n}\int_{\mathbb{R}^n}\xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n}\big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 - \xi_i^2|\widehat{u}_i(\xi)|^2 - \xi_j^2|\widehat{u}_j(\xi)|^2\big]d\xi \\
&\geqslant \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n}\big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2\big]d\xi \\
&\geqslant \frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\xi_i^2|\widehat{u}_j(\xi)|^2 d\xi = \frac{1}{2}\sum_{i,j=1}^{n}\Big\|\frac{\partial u_j}{\partial x_i}\Big\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

□

Problem 3. 1. Let d_r denote the dilation operator defined by $d_r f(x) = f\big(\dfrac{x}{r}\big)$. Show that

$$
\mathscr{F}(d_r f) = r^n d_{1/r}\mathscr{F}(f) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{0.1}
$$

2. In some occasions (especially in engineering applications), the Fourier transform and inverse Fourier transform of a (Schwartz) function f are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi x\cdot\xi} dx \qquad \text{and} \qquad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{i2\pi x\cdot\xi} d\xi\,.$$

Show that under this definition, $\check{\widehat{f}} = \widehat{\check{f}} = f$ for all $f \in \mathscr{S}(\mathbb{R}^n)$. Note that you can use the Fourier Inversion Formula that we derive in class.

Proof. Let $\mathscr{F}$ denote the Fourier transform operator that we used in class, and $\widehat{\ }$ be the Fourier transform operator in this problem.

1. Let d_r denote the dilation operator define by $(d_r f)(x) = f(rx)$. By the change of variables formula,

$$\begin{aligned}\mathscr{F}(d_r f)(\xi) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} (d_r f)(x) e^{-ix\cdot\xi}\, dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(r^{-1}x) e^{-ix\cdot\xi}\, dx \\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y) e^{-iry\cdot\xi} r^n\, dy = \frac{r^n}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y) e^{-iy\cdot(r\xi)}\, dy \\ &= r^n \mathscr{F}(f)(r\xi) = r^n \big[d_{\frac{1}{r}}\mathscr{F}(f)\big](\xi)\end{aligned}$$

so that (0.1) is established.

2. Replacing f by $d_{1/r} f$ in (0.1) implies that

$$\mathscr{F}(f) = \mathscr{F}\big(d_r d_{\frac{1}{r}} f\big) = r^n d_{\frac{1}{r}} \mathscr{F}\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond$}$$

Similarly, $\mathscr{F}^*(d_r f) = r^n d_{\frac{1}{r}} \mathscr{F}^*(f)$ so that

$$\mathscr{F}^*(f) = r^n d_{\frac{1}{r}} \mathscr{F}^*\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond\diamond$}$$

Note that

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx = \sqrt{2\pi}^n \mathscr{F}(f)(2\pi\xi) = \sqrt{2\pi}^n \big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi) \\ &= \frac{1}{\sqrt{2\pi}^n}(2\pi)^n \big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi} f)(\xi)\end{aligned}$$

and

$$\check{f}(\xi) = \widehat{f}(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi} f)(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi} f)(\xi)\,.$$

Therefore, ($\diamond$) implies that

$$\begin{aligned}\check{\widehat{f}}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi}\widehat{f})(\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*\Big(\frac{1}{\sqrt{2\pi}^n} d_{2\pi}\mathscr{F}(d_{2\pi} f)\Big)(\xi) \\ &= \mathscr{F}^*\big((2\pi)^{-n} d_{2\pi}\mathscr{F}(d_{2\pi} f)\big)(\xi) = \mathscr{F}^*(\mathscr{F} f)(\xi) = f(\xi)\,.\end{aligned}$$

Similarly, ($\diamond\diamond$) implies that

$$\widehat{\check{f}}(\xi) = \mathscr{F}\big((2\pi)^{-n} d_{2\pi}\mathscr{F}^*(d_{2\pi} f)\big)(\xi) = \mathscr{F}(\mathscr{F}^* f)(\xi) = f(\xi)\,.$$

□

Problem 4. 1. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous integrable function such that $\widehat{f}$ is also integrable. Show that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) \cos[(x-y)\xi]\, dy \Big)\, d\xi \qquad \forall\, x \in \mathbb{R}\,.$$

2. If in addition to condition in 1, f is an even function. Show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty f(y) \cos(x\xi) \cos(y\xi)\, dy \Big)\, d\xi\,.$$

3. If in addition to condition in 1, f is an odd function. Show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty f(y) \sin(x\xi) \sin(y\xi)\, dy \Big)\, d\xi\,.$$

4. For a function $g : [0, \infty) \to \mathbb{C}$ satisfying $\displaystyle\int_0^\infty \big|g(x)\big|\, dx < \infty$, the Fourier cosine transform and the Fourier sine transform of g, denoted by $\mathscr{F}_{\cos}[g]$ and $\mathscr{F}_{\sin}[g]$ respectively, are functions defined by

$$\mathscr{F}_{\cos}[g](\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \cos(y\xi)\, dy \quad \text{and} \quad \mathscr{F}_{\sin}[g](\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \sin(y\xi)\, dy\,.$$

(a) Show that if $\mathscr{F}_{\cos}[g] \in L^1(\mathbb{R})$, then

$$g(x) = \mathscr{F}_{\cos}\big[\mathscr{F}_{\cos}[g]\big](x) \qquad \text{whenever } x \in [0,\infty) \text{ and } g \text{ is continuous at } x,$$

or equivalently,

$$g(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty g(y) \cos(y\xi)\, dy \Big) \cos(x\xi)\, d\xi$$

whenever $x \in [0, \infty)$ and g is continuous at x.

(b) Show that if $\mathscr{F}_{\sin}[g] \in L^1(\mathbb{R})$, then

$$g(x) = \mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) \qquad \text{whenever } x \in [0,\infty) \text{ and } g \text{ is continuous at } x,$$

or equivalently,

$$g(x) = \frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty g(y) \sin(y\xi)\, dy \Big) \sin(x\xi)\, d\xi$$

whenever $x \in (0, \infty)$ and g is continuous at x.

Hint of 4: Consider the even or odd extension of g, and apply conclusions in 2 and 3.

Proof. 1. Let f be a continuous integrable function such that $\widehat{f}$ is also integrable. Then $\check{f}$ is also integrable; thus the Fourier inversion formula implies that

$$f(x) = \check{\widehat{f}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{-iy\xi}\, dy \Big) e^{ix\xi}\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{i(x-y)\xi}\, dy \Big)\, d\xi$$

and

$$f(x) = \widehat{\check{f}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{iy\xi}\, dy \Big) e^{-ix\xi}\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y) e^{-i(x-y)\xi}\, dy \Big)\, d\xi$$

whenever f is continuous at x. Therefore, if f is continuous at x, then

$$\begin{aligned} f(x) &= \frac{1}{2}\Big[\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)e^{i(x-y)\xi}\,dy\Big)\,d\xi + \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)e^{-i(x-y)\xi}\,dy\Big)\,d\xi\Big] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\frac{e^{i(x-y)\xi}+e^{-i(x-y)\xi}}{2}\,dy\Big)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\cos[(x-y)\xi]\,dy\Big)\,d\xi\,. \end{aligned}$$

We note that by the sum and difference of angles identities, the identity above implies that

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\big[\cos(x\xi)\cos(y\xi)+\sin(x\xi)\sin(y\xi)\big]\,dy\Big)\,d\xi\,. \tag{0.2}$$

2. If f is an even function, then $\int_{\mathbb{R}} f(y)\sin(x\xi)\sin(y\xi)\,dy = 0$; thus (0.2) shows that if f is continuous at x,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\cos(x\xi)\cos(y\xi)\,dy\Big)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(2\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi\,. \end{aligned}$$

Note that the inner integral is an even function of ξ, so

$$\begin{aligned} f(x) &= \frac{2}{2\pi}\int_0^\infty\Big(2\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi \\ &= \frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\cos(y\xi)\,dy\Big)\cos(x\xi)\,d\xi\,. \end{aligned}$$

3. If f is an odd function, then $\int_{\mathbb{R}} f(y)\cos(x\xi)\cos(y\xi)\,dy = 0$; thus (0.2) shows that if f is continuous at x,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(y)\sin(x\xi)\sin(y\xi)\,dy\Big)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(2\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi\,. \end{aligned}$$

Note that the inner integral is an odd function of ξ, so

$$\begin{aligned} f(x) &= \frac{2}{2\pi}\int_0^\infty\Big(2\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi \\ &= \frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(y)\sin(y\xi)\,dy\Big)\sin(x\xi)\,d\xi\,. \end{aligned}$$

4. Suppose that $g : [0,\infty) \to \mathbb{C}$ is integrable.

   (a) Let $f : \mathbb{R} \to \mathbb{C}$ be defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \geqslant 0\,, \\ g(-x) & \text{if } x < 0\,. \end{cases}$$

Then f is an even function and is integrable on $\mathbb{R}$. Moreover,

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-iy\xi}\, dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) \big[\cos(y\xi) - i\sin(y\xi)\big]\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\cos(y\xi)\, dy - i\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\sin(y\xi)\, dy\,.\end{aligned}$$

By the definition of f,

$$\begin{aligned}\int_{\mathbb{R}} f(y)\cos(y\xi)\, dy &= \int_0^\infty f(y)\cos(y\xi)\, dy + \int_{-\infty}^0 f(y)\cos(y\xi)\, dy \\ &= \int_0^\infty g(y)\cos(y\xi)\, dy + \int_{-\infty}^0 g(-y)\cos(y\,xi)\, dy \\ &= \int_0^\infty g(y)\cos(y\xi)\, dy + \int_\infty^0 g(y)\cos(-y\xi)\, d(-y) \\ &= 2\int_0^\infty g(y)\cos(y\xi)\, dy = \sqrt{2\pi}\mathscr{F}_{\cos}[g](\xi)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} f(y)\sin(y\xi)\, dy &= \int_0^\infty f(y)\sin(y\xi)\, dy + \int_{-\infty}^0 f(y)\sin(y\xi)\, dy \\ &= \int_0^\infty g(y)\sin(y\xi)\, dy + \int_{-\infty}^0 g(-y)\sin(y\,xi)\, dy \\ &= \int_0^\infty g(y)\sin(y\xi)\, dy + \int_\infty^0 g(y)\sin(-y\xi)\, d(-y) = 0\,;\end{aligned}$$

thus $\widehat{f} = \mathscr{F}_{\cos}[g]$ which implies that $\widehat{f} \in L^1(\mathbb{R})$. On the other hand, $\check{f}(\xi) = \widehat{f}(-\xi) = \mathscr{F}_{\cos}[g](\xi)$; thus the Fourier inversion formula implies that

$$\mathscr{F}_{\cos}\big[\mathscr{F}_{\cos}[g]\big](x) = \check{\widehat{f}}(x) = f(x)$$

whenever f is continuous at x. In particular, if $x \in [0,\infty)$ and g is continuous at x, then f is continuous at x and $f(x) = g(x)$ which imply that

$$\mathscr{F}_{\cos}\big[\mathscr{F}_{\cos}[g]\big](x) = g(x) \quad \text{whenever } x \in (0,\infty) \text{ and } g \text{ is continuous at } x.$$

(b) Let $f : \mathbb{R} \to \mathbb{C}$ be defined by

$$f(x) = \begin{cases} g(x) & \text{if } x > 0\,, \\ -g(-x) & \text{if } x < 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f is an odd function and is integrable on $\mathbb{R}$. Moreover,

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-iy\xi}\, dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) \big[\cos(y\xi) - i\sin(y\xi)\big]\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\cos(y\xi)\, dy - i\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\sin(y\xi)\, dy\,.\end{aligned}$$

By the definition of f,

$$\begin{aligned}\int_{\mathbb{R}} f(y)\cos(y\xi)\,dy &= \int_0^\infty f(y)\cos(y\xi)\,dy + \int_{-\infty}^0 f(y)\cos(y\xi)\,dy\\ &= \int_0^\infty g(y)\cos(y\xi)\,dy - \int_{-\infty}^0 g(-y)\cos(y\,xi)\,dy\\ &= \int_0^\infty g(y)\cos(y\xi)\,dy - \int_\infty^0 g(y)\cos(-y\xi)\,d(-y) = 0\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} f(y)\sin(y\xi)\,dy &= \int_0^\infty f(y)\sin(y\xi)\,dy + \int_{-\infty}^0 f(y)\sin(y\xi)\,dy\\ &= \int_0^\infty g(y)\sin(y\xi)\,dy - \int_{-\infty}^0 g(-y)\sin(y\,xi)\,dy\\ &= \int_0^\infty g(y)\sin(y\xi)\,dy - \int_\infty^0 g(y)\sin(-y\xi)\,d(-y)\\ &= 2\int_0^\infty g(y)\sin(y\xi)\,dy = \sqrt{2\pi}\mathscr{F}_{\sin}[g](\xi)\,;\end{aligned}$$

thus $\widehat{f} = -i\mathscr{F}_{\sin}[g]$ which implies that $\widehat{f} \in L^1(\mathbb{R})$. On the other hand, $\check{\widehat{f}}(\xi) = \widehat{f}(-\xi) = i\mathscr{F}_{\sin}[g](\xi)$; thus the Fourier inversion formula implies that

$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) = -i\mathscr{F}_{\sin}\big[i\mathscr{F}_{\sin}[g]\big](x) = \check{\widehat{f}}(x) = f(x)$$

whenever f is continuous at x. In particular, if $x \in (0,\infty)$ and g is continuous at x, then f is continuous at x and $f(x) = g(x)$ which imply that

$$\mathscr{F}_{\sin}\big[\mathscr{F}_{\sin}[g]\big](x) = g(x) \qquad \text{whenever } x \in (0,\infty) \text{ and } g \text{ is continuous at } x. \qquad \square$$

Problem 5. Suppose that $f \in L^1(\mathbb{R})$ is continuous and $\widehat{f}(\xi) = \dfrac{\ln(1+\xi^2)}{\xi^2}$. Find $f(0)$ and $\displaystyle\int_{-\infty}^\infty f(x)\,dx$.

Solution. First we claim that $\widehat{f} \in L^1(\mathbb{R})$. To see this, note that $\widehat{f} \geqslant 0$ so that

$$\begin{aligned}\int_{\mathbb{R}} \big|\widehat{f}(\xi)\big|\,d\xi &= \int_{-\infty}^\infty \frac{\ln(1+\xi^2)}{\xi^2}\,d\xi = \frac{-\ln(1+\xi^2)}{\xi}\Big|_{\xi=-\infty}^{\xi=\infty} + \int_{-\infty}^\infty \frac{1}{\xi}\frac{2\xi}{1+\xi^2}\,d\xi\\ &= 2\int_{-\infty}^\infty \frac{1}{1+\xi^2}\,d\xi = 2\arctan\xi\Big|_{\xi=-\infty}^{\xi=\infty} = 2\pi\,.\end{aligned}$$

Therefore, we can apply the Fourier inversion formula to obtain that

$$f(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\xi\cdot 0}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\ln(1+\xi^2)}{\xi^2}\,d\xi = \frac{2\pi}{\sqrt{2\pi}} = \sqrt{2\pi}\,.$$

Moreover, by the definition and the property of the Fourier transform,

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\xi\to 0}\sqrt{2\pi}\widehat{f}(\xi) = \sqrt{2\pi}\lim_{t\to 0^+}\frac{\ln(1+t)}{t} = \sqrt{2\pi}\,. \qquad \square$$