Exercise Problem Sets 8

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Problem 1. A family of functions $\{\varphi_n \in \mathscr{C}(\mathbb{T}) \,|\, n \in \mathbb{N}\}$ is called an approximation of the identity if

(1) $\varphi_n(x) \geqslant 0$;

(2) $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$ for every $n \in \mathbb{N}$;

(3) $\displaystyle\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx = 0$ for every $\delta > 0$, here we identify $\mathbb{T}$ with the interval $[-\pi, \pi]$.

Show that if $\{\varphi_n\}_{n=1}^\infty$ is an approximation of the identity and $f \in \mathscr{C}(\mathbb{T})$, then $\{\varphi_n \star f\}_{n=1}^\infty$ converges uniformly to f as $n \to \infty$.

Remark: By the definition above, we find that the Fejér kernel $\{F_n\}_{n=1}^\infty$ is an approximation of the identity.

Proof. W.L.O.G., we may assume that $f \not\equiv 0$. By the definition of the convolution,

$$\begin{aligned}\big|(\varphi_n \star f)(x) - f(x)\big| &= \int_{\mathbb{T}} \varphi_n(x-y) f(y)\,dy - f(x) \\ &= \int_{\mathbb{T}} \varphi_n(x-y)\big(f(x) - f(y)\big)dy\,,\end{aligned}$$

where we use (2) of the definition above to obtain the last equality. Now given $\varepsilon > 0$. Since $f \in \mathscr{C}(\mathbb{T})$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $|x - y| < \delta$. Therefore,

$$\begin{aligned}&|(\varphi_n \star f)(x) - f(x)| \\ &\quad\leqslant \int_{|x-y|<\delta} \varphi_n(x-y)\big|f(x) - f(y)\big|dy + \int_{\delta \leqslant |x-y|} \varphi_n(x-y)\big|f(x) - f(y)\big|dy \\ &\quad\leqslant \frac{\varepsilon}{2}\int_{\mathbb{T}} \varphi_n(x-y)\,dy + 2\max_{\mathbb{T}}|f| \int_{\delta \leqslant |z| \leqslant \pi} \varphi_n(z)\,dz\,.\end{aligned}$$

By (3) of the definition above, there exists $N > 0$ such that if $n \geqslant N$,

$$\int_{\delta \leqslant |z| \leqslant \pi} \varphi_n(z)\,dx < \frac{\varepsilon}{4\max_{\mathbb{T}}|f|}\,.$$

Therefore, for $n \geqslant N$, $\big|(\varphi_n \star f)(x) - f(x)\big| < \varepsilon$ for all $x \in \mathbb{T}$. □

Problem 2. In this problem we show that the collection of trigonometric polynomials $\mathscr{P}(\mathbb{T})$ (defined in Corollary 7.85 in the lecture note) is dense in $\mathscr{C}(\mathbb{T})$ in another way. Complete the following.

1. Let $\varphi_n(x) = c_n(1 + \cos x)^n$, where c_n is chosen so that $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$. Show that

$$c_n = \frac{2^{n-1}}{\pi}\frac{(n!)^2}{(2n)!}\,.$$

2. Show that for each $0 < \delta < \pi$,

$$\lim_{n\to\infty} \int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx = 0\,.$$

In other words, $\{\varphi_n\}_{n=1}^\infty$ is an approximation of the identity. Therefore, Problem 1 shows that $\{\varphi_n \star f\}_{n=1}^\infty$ converges uniformly to f as $n \to \infty$ if $f \in \mathscr{C}(\mathbb{T})$.

3. Show that $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$.

Proof. 1. Let $\varphi_n(x) = c_n(1+\cos x)^n$, where c_n is chosen so that $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$. First we note that by Wallis's formula,

$$\begin{aligned}\int_{-\pi}^{\pi} (1+\cos x)^n\,dx &= 2^n \int_{-\pi}^{\pi} \Big(\frac{1+\cos x}{2}\Big)^n dx = 2^n \int_{-\pi}^{\pi} \cos^{2n}\frac{x}{2}\,dx = 2^{n+1}\int_0^{\pi}\cos^{2n}\frac{x}{2}\,dx \\ &= 2^{n+2}\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = 2^{n+2}\frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2} = \frac{\pi(2n)!}{2^{n-1}(n!)^2}\,.\end{aligned}$$

Therefore,

$$1 = \int_{\mathbb{T}} \varphi_n(x)\,dx = c_n \int_{-\pi}^{\pi} (1+\cos x)^n\,dx = \frac{\pi(2n)!}{2^{n-1}(n!)^2}c_n$$

which implies that

$$c_n = \frac{2^{n-1}}{\pi}\frac{(n!)^2}{(2n)!}\,.$$

2. Now $\{\varphi_n\}_{n=1}^\infty$ is clearly non-negative and satisfies (2) of the definition of an approximation of identity (given in Problem 1) for all $n \in \mathbb{N}$. Let $\delta > 0$ be given.

$$\int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx \leqslant \int_{\delta\leqslant|x|\leqslant\pi} c_n(1+\cos\delta)^n dx \leqslant 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n\frac{(n!)^2}{(2n)!}\,.$$

By Stirling's formula $\displaystyle\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}n^n e^{-n}} = 1$,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx &\leqslant \lim_{n\to\infty} 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n\frac{\big(\sqrt{2\pi n}n^n e^{-n}\big)^2}{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}} \\ &= \lim_{n\to\infty}\sqrt{\pi n}\Big(\frac{1+\cos\delta}{2}\Big)^n = 0\,;\end{aligned}$$

thus

$$\lim_{n\to\infty}\int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx = 0\,.$$

So $\{\varphi_n\}_{n=1}^\infty$ is an approximation of the identity. By the result in Problem 1, $\{\varphi_k\star f\}_{k=1}^\infty$ converges uniformly to f if $f \in \mathscr{C}(\mathbb{T})$.

3. It suffices to show that $\varphi_n \star f$ is a trigonometric polynomial for each $n \in \mathbb{N}$. Nevertheless,

$$\begin{aligned}(\varphi_n \star f)(x) &= \int_{\mathbb{T}} c_n\big[1+\cos(x-y)\big]^n f(y)\,dy = c_n\int_{-\pi}^{\pi}(1+\cos x\cos y+\sin x\sin y)^n f(y)\,dy \\ &= c_n\int_{-\pi}^{\pi}\sum_{0\leqslant k,\ell\leqslant n,k+\ell\leqslant n}\frac{n!}{k!\ell!(n-k-\ell)!}\cos^k x\cos^k y\sin^\ell x\sin^\ell y f(y)\,dy \\ &= \sum_{0\leqslant k,\ell\leqslant n,k+\ell\leqslant n} A_{n,k,\ell}\cos^k x\sin^\ell x\,,\end{aligned}$$

where

$$A_{n,k,\ell} = c_n \int_{-\pi}^{\pi} \cos^k y \sin^\ell y f(y)\, dy\,.$$

The final conclusion follows because the function $y = \cos^k x \sin^\ell x$ belongs to $\mathscr{P}_{k+\ell}(\mathbb{T})$. □