# Exercise Problem Sets 3

Mar. 05. 2022

**Problem 1.** Define a set $S \subseteq [0,1] \times [0,1]$ by

$$S = \Big\{ \big(\frac{p}{m}, \frac{k}{m}\big) \in [0,1] \times [0,1] \,\Big|\, m, p, k \in \mathbb{N}\,, \gcd(m,p) = 1 \text{ and } 1 \leqslant k \leqslant m-1 \Big\}.$$

Show that

$$\int_0^1 \Big( \int_0^1 \mathbf{1}_S(x,y)\,dy \Big) dx = \int_0^1 \Big( \int_0^1 \mathbf{1}_S(x,y)\,dx \Big) dy = 0$$

but $\mathbf{1}_S$ is not Riemann integrable on $[0,1] \times [0,1]$.

*Proof.* Note that for each $x \in [0,1]$, then $\mathbf{1}_S(x,y) \neq 0$ for only finitely many $y \in [0,1]$. Therefore, for each $x \in [0,1]$, $\mathbf{1}_S(x,\cdot)$ is Riemann integrable on $[0,1]$ and

$$\int_0^1 \mathbf{1}_S(x,y)\,dy = 0\,.$$

Similarly, for each $y \in [0,1]$, then $\mathbf{1}_S(x,y) \neq 0$ for only finitely many $x \in [0,1]$; thus for each $y \in [0,1]$, $\mathbf{1}_S(x,\cdot)$ is Riemann integrable on $[0,1]$ and

$$\int_0^1 \mathbf{1}_S(x,y)\,dx = 0\,.$$

Therefore,

$$\int_0^1 \Big( \int_0^1 \mathbf{1}_S(x,y)\,dy \Big) dx = \int_0^1 \Big( \int_0^1 \mathbf{1}_S(x,y)\,dx \Big) dy = 0\,.$$

However, for each partition $\mathcal{P}$ of $[0,1] \times [0,1]$, we have $\Delta \cap S \neq \varnothing$ for all $\Delta \in \mathcal{P}$; thus $U(\mathbf{1}_S, \mathcal{P}) = 1$ for all partition $\mathcal{P}$ of $[0,1] \times [0,1]$. Therefore,

$$\overline{\int}_{A \times B} \mathbf{1}_S(x,y)\,dy = 1$$

which, by the Fubini Theorem, implies that $\mathbf{1}_S$ is not Riemann integrable on $[0,1] \times [0,1]$. □

**Problem 2.** Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

1. Show that $\displaystyle\int_0^1 f(x,y)\,dx = 0$ for all $y \in \big[0, \frac{1}{2}\big)$.
2. Show that $\displaystyle\int_0^1 f(x,y)\,dy = 0$ for all $x \in [0,1)$.
3. Justify if the iterated (improper) integrals $\displaystyle\int_0^1 \int_0^1 f(x,y) dx dy$ and $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dydx$ are identical.

*Proof.* 1. Since $f(x,0) = 0$ for all $x \in [0,1]$, we have $\int_0^1 f(x,0)\,dx = 0$. Suppose that $y \in \big(0, \frac{1}{2}\big)$. Then $y \in [2^{-n}, 2^{-n+1})$ for a unique natural number $n \geqslant 2$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } x \in [2^{-n}, 2^{-n+1})\,, \\ -2^{2n-1} & \text{if } x \in [2^{-n+1}, 2^{-n+2})\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dx &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n+1},2^{-n+2})} -2^{2n-1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n-1}(2^{-n+2} - 2^{-n+1}) = 0\,. \end{aligned}$$

2. Since $f(0,y)$ for all $y \in [0,1]$, we have $\int_0^1 f(0,y)\,dy = 0$. Suppose tat $x \in (0,1)$. Then $x \in [2^{-n}, 2^{-n+1})$ for a unique $n \in \mathbb{N}$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } y \in [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } y \in [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dy &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n-1},2^{-n})} -2^{2n+1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n+1}(2^{-n} - 2^{-n-1}) = 0\,. \end{aligned}$$

3. By 2, we immediately conclude that

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = 0\,.$$

On the other hand, note that if $y \in \big[\frac{1}{2}, 1\big)$, then $f(x,y) = \begin{cases} 4 & \text{if } x \in \big[\frac{1}{2}, 1\big)\,, \\ 0 & \text{otherwise}\,, \end{cases}$ so that

$$\int_0^1 f(x,y)\,dx = \int_{\frac{1}{2}}^1 4\,dx = 2\,.$$

Therefore,

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}} \int_0^1 f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^1 \int_0^1 f(x,y)\,dx\,dy = \int_{\frac{1}{2}}^1 2\,dy = 1$$

which shows that $\int_0^1 \int_0^1 f(x,y)dxdy \neq \int_0^1 \int_0^1 f(x,y)\,dydx$ for this particular $f$. □

**Problem 3** (The multiple integral version of Theorem 6.65 in the lecture note)**.** Let $A$ be a closed rectangle in $\mathbb{R}^n$, and $f_k : A \to \mathbb{R}$ be a decreasing sequence of bounded functions. Show (without applying Theorem 6.69 and 6.70 in the lecture note) that if $\lim\limits_{k\to\infty} f_k(x) = 0$ for all $x \in A$, then

$$\lim_{k\to\infty} \underline{\int_A} f_k(x)\,dx = 0\,.$$

Conclude the Monotone Convergence Theorem (Theorem 6.69 in the lecture note) and the Bounded Convergence Theorem (Theorem 6.70 in the lecture note) using the this conclusion of convergence.

**Problem 4.** Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be Riemann measurable sets, and $f : A\times B \to \mathbb{R}$ be non-negative, uniformly continuous and integrable on $A \times B$. Define $F(x) = \displaystyle\int_B f(x,y)\,dy$.

1. Show that if $B$ is bounded, then $F : A \to \mathbb{R}$ is continuous. How about if $B$ is not bounded?

2. Let $f$ have the additional property that for each $\varepsilon > 0$, there exists $N > 0$ such that

$$\Big|\int_{B\cap B(0,k)} (f\wedge k)(x,y)\,dy - \int_B f(x,y)\,dy\Big| < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

Show that $F$ is continuous on $A$. In particular, show that if $f(x,y) \leqslant g(y)$ for all $(x,y) \in A\times B$, and $g$ is integrable on $B$, then $F$ is continuous.

*Proof.* 1. If $B$ is bounded, then $B$ has volume. Let $\varepsilon > 0$ be given. By the uniform continuity of $f$, there exists $\delta > 0$ such that

$$\big|f(x_1,y_1) - f(x_2,y_2)\big| < \frac{\varepsilon}{\nu(B)+1} \qquad \forall\,\big|(x_1,y_1)-(x_2,y_2)\big| < \delta \text{ and } x_1,x_2 \in A, y_1,y_2 \in B\,.$$

Therefore, if $|x_1 - x_2| < \delta$ and $x_1, x_2 \in A$,

$$\begin{aligned}\big|F(x_1) - F(x_2)\big| &= \Big|\int_B \big[f(x_1,y) - f(x_2,y)\big]\,dy\Big| \leqslant \int_B \big|f(x_1,y) - f(x_2,y)\big|\,dy \\ &\leqslant \int_B \frac{\varepsilon}{\nu(B)+1}\,dx \leqslant \frac{\varepsilon\nu(B)}{\nu(B)+1} < \varepsilon\,.\end{aligned}$$

This implies that $F$ is uniformly continuous on $A$.

If $B$ is unbounded, then the argument above does not apply. In fact, consider the case

$$f(x,y) = \frac{\sqrt{x}}{1+x^2y^2}\,, \qquad A = [0,1] \quad\text{and}\quad B = \mathbb{R}\,.$$

Then $f$ is non-negative and uniformly continuous on $A \times B$ (by Exercise Problem **??**). Note that $F(0) = 0$ while if $x > 0$,

$$F(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{-\infty}^{\infty} \frac{\sqrt{x}}{1+x^2y^2}\,dy = \frac{\sqrt{x}}{x}\arctan(xy)\Big|_{y=-\infty}^{y=\infty} = \frac{\pi}{\sqrt{x}}\,.$$

Therefore, the Tonelli Theorem implies that

$$\int_{A\times B} f(x,y)\,d(x,y) = \int_A\Big(\int_B f(x,y)\,dy\Big)\,dx = \int_0^1 \frac{\pi}{\sqrt{x}}\,dx = 2\pi < \infty$$

which shows that $f$ is integrable on $A \times B$. However, $F$ is not continuous at $x = 0$.

2. Let $\varepsilon > 0$ be given. Since $f$ has the property mentioned above, there exists $N > 0$ such that

$$\Big| \int_{B \cap B(0,k)} (f \wedge k)(x, y)\, dy - \int_B f(x, y)\, dy \Big| < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the uniform continuity of $f$ on $A \times B$, there exists $\delta > 0$ such that

$$\big| f(x_1, y_1) - f(x_2, y_2) \big| < \frac{\varepsilon}{3} \qquad \forall\, \big|(x_1, y_1) - (x_2, y_2)\big| < \delta \text{ and } x_1, x_2 \in A, y_1, y_2 \in B\,.$$

Suppose that $|x_1 - x_2| < \delta$, $x_1, x_2 \in A$ and $y \in B$.

(a) If $f(x_1, y)$ and $f(x_2, y)$ are both not greater than $N$, then

$$\big|(f \wedge N)(x_1, y) - (f \wedge N)(x_2, y)\big| = \big|f(x_1, y) - f(x_2, y)\big| < \varepsilon\,.$$

(b) If $f(x_1, y)$ and $f(x_2, y)$ are both greater than $N$, then

$$\big|(f \wedge N)(x_1, y) - (f \wedge N)(x_2, y)\big| = |N - N| = 0\,.$$

(c) If one and only one of $f(x_1, y)$ and $f(x_2, y)$ is greater than $N$, then

$$\big|(f \wedge N)(x_1, y) - (f \wedge N)(x_2, y)\big| < \big|f(x_1, y) - f(x_2, y)\big| < \varepsilon\,.$$

Case (a), (b) and (c) show that

$$\big|(f \wedge N)(x_1, y) - (f \wedge N)(x_2, y)\big| < \frac{\varepsilon}{3\nu(B(0, N))} \quad \forall\, |x_1 - x_2| < \delta, x_1, x_2 \in A \text{ and } y \in B\,.$$

Therefore, if $x_1, x_2 \in A$ and $|x_1 - x_2| < \delta$,

$$\begin{aligned}
\big|F(x_1) - F(x_2)\big| &\leqslant \Big| \int_{B \cap B(0,N)} (f \wedge N)(x_1, y)\, dy - \int_B f(x_1, y)\, dy \Big| \\
&\quad + \Big| \int_{B \cap B(0,N)} (f \wedge N)(x_2, y)\, dy - \int_B f(x_2, y)\, dy \Big| \\
&\quad + \Big| \int_{B \cap B(0,N)} (f \wedge N)(x_1, y)\, dy - \int_{B \cap B(0,N)} (f \wedge N)(x_2, y)\, dy \Big| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_{B \cap B(0,N)} \big|(f \wedge N)(x_1, y) - (f \wedge N)(x_2, y)\big|\, dy \leqslant \varepsilon\,.
\end{aligned}$$

This implies that $F$ is uniformly continuous on $A$.

Now suppose that $f(x, y) \leqslant g(y)$ for all $(x, y) \in A \times B$, and $g$ is integrable on $B$. Then

$$\lim_{k \to \infty} \int_{B \cap B(0,k)} (g \wedge k)(y)\, dy = \int_B g(y)\, dy\,;$$

thus there exists $N > 0$ such that

$$\Big| \int_{B \cap B(0,k)} (g \wedge k)(y)\, dy - \int_B g(y)\, dy \Big| < \varepsilon \qquad \text{whenever} \quad k \geqslant N\,.$$

Therefore, for all $k \geqslant N$ and $x \in A$,

$$
\begin{aligned}
\Big| \int_{B \cap B(0,k)} & (f \wedge k)(x,y)\, dy - \int_B f(x,y)\, dy \Big| \\
&\leqslant \Big| \int_{B \cap B(0,k)} (f \wedge k)(x,y)\, dy - \int_{B \cap B(0,k)} f(x,y)\, dy \Big| + \int_{B \cap B(0,k)^{\complement}} f(x,y)\, dy \\
&\leqslant \int_{B \cap B(0,k)} \big| (f \wedge k)(x,y) - f(x,y) \big|\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\leqslant \int_{\{y \in B \cap B(0,k) \,|\, f(x,y) > k\}} \big[ f(x,y) - k \big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\leqslant \int_{\{y \in B \cap B(0,k) \,|\, g(y) > k\}} \big[ g(y) - k \big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\leqslant \int_{B \cap B(0,k)} \big[ g(y) - (g \wedge k)(y) \big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&= \int_B g(y)\, dy - \int_{B \cap B(0,k)} (g \wedge k)(y)\, dy < \varepsilon\,.
\end{aligned}
$$

This shows that $f$ satisfies the condition mentioned in 2, so $F$ is continuous on $A$. □

**Problem 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be a Riemann measurable function, and $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y) \cos(x - y)\, dy$$

whenever the integral exists. Show that if the function $f$ is integrable, then $F$ is defined on $\mathbb{R}$ and is differentiable on $\mathbb{R}$ with derivative

$$F'(x) = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial x} \cos(x - y)\, dy = -\int_{\mathbb{R}} f(y) \sin(x - y)\, dy\,.$$

*Proof.* Let $x \in \mathbb{R}$ be given. Since $f$ is Riemann measurable, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = f(y)\cos(x - y)$ is Riemann measurable and $|g(y)| \leqslant |f(y)|$ for all $y \in \mathbb{R}$. Since $f$ is integrable, the comparison test implies that $g$ is integrable. Therefore, $F$ is defined everywhere on $\mathbb{R}$.

Let $\{h_k\}_{k=1}^{\infty}$ be a non-zero sequence with limit 0. Define

$$g_k(y) = f(y) \frac{\cos(x + h_k - y) - \cos(x - y)}{h_k}\,.$$

Then for all $y \in \mathbb{R}$, $\lim\limits_{k \to \infty} g_k(y) = f(y) \dfrac{\partial}{\partial x}(cos(x - y)) = -f(y)\sin(x - y)$.

Since $\left| \dfrac{d}{dx} \cos x \right| \leqslant 1$, the Mean Value Theorem implies that

$$\big| \cos(x + h_k - y) - \cos(x - y) \big| \leqslant |h_k|\,.$$

Therefore,

$$\big| g_k(y) \big| \leqslant \big| f(y) \big| \qquad \forall\, x \in \mathbb{R}\,.$$

Since $f$ is integrable on $\mathbb{R}$, $|f|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k \to \infty} \frac{F(x + h_k) - F(x)}{h_k} = \lim_{k \to \infty} \int_{\mathbb{R}} g_k(y)\, dy = -\int_{\mathbb{R}} f(x) \sin(x - y)\, dy\,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^{\infty}$ with limit 0, the limit

$$\lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dy$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dy\,. \qquad \square$$

**Problem 6.** Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable Riemann measurable function, and $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y)\cos(xy)\,dy$$

(which exists for all $x\in\mathbb{R}$ since $f$ is integrable). Show that if the function $g(x)=xf(x)$ is integrable, then $F$ is differentiable on $\mathbb{R}$ and

$$F'(y) = \int_{\mathbb{R}} f(x)\frac{\partial}{\partial y}\cos(xy)\,dy = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dy\,.$$

*Proof.* Let $y\in\mathbb{R}$ be given, and $\{h_k\}_{k=1}^{\infty}$ be a non-zero sequence with limit 0. Define

$$g_k(x) = f(x)\frac{\cos(x(y+h_k))-\cos(xy)}{h_k}\,.$$

Then for all $x\in\mathbb{R}$, $\lim\limits_{k\to\infty} g_k(x) = f(x)\dfrac{\partial}{\partial y}(cos(xy)) = -xf(x)\sin(xy)$.

Since $\left|\dfrac{d}{dy}\cos x\right| \leqslant 1$, the Mean Value Theorem implies that

$$\big|\cos(x(y+h_k))-\cos(xy)\big| \leqslant |xh_k|\,.$$

Therefore,

$$\big|g_k(x)\big| \leqslant \big|xf(x)\big| = \big|g(x)\big| \qquad \forall\, x\in\mathbb{R}\,.$$

Since $g$ is integrable on $\mathbb{R}$, $|g|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty} \frac{F(y+h_k)-F(y)}{h_k} = \lim_{k\to\infty}\int_{\mathbb{R}} h_k(x)\,dy = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dy\,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^{\infty}$ with limit 0, the limit

$$\lim_{k\to\infty} \frac{F(y+h_k)-F(y)}{h_k} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dy$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0} \frac{F(y+h)-F(y)}{h} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dy\,. \qquad \square$$

**Problem 7.** Let $f(x,y) = \begin{cases} \dfrac{e^{-xy}\sin y}{y} & \text{if } y\neq 0\,, \\ 1 & \text{if } y=0\,. \end{cases}$.

1. Show that $f_x(x,y)$ is continuous everywhere, and show that $f(x,\cdot)$ is integrable on $[0,\infty)$ for all $x>0$.

2. Define $F(x)=\displaystyle\int_0^\infty f(x,y)\,dy$ for $x>0$. Show that $F'(x)=-\dfrac{1}{x^2+1}$.

3. Show that $F(x)=\dfrac{\pi}{2}-\tan^{-1}x$ if $x>0$, and conclude that

$$\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}\,.$$

*Proof.* 1. Note that if $y\neq 0$, $f_x(x,y)=e^{-xy}\sin y$ while $f_x(x,0)=0$. Clearly $f_x$ is continuous on $\mathbb{R}^2$ except perhaps on the $x$-axis. On the other hand, since $\displaystyle\lim_{(x,y)\to(a,0)} f(x,y)=0$, we conclude that $f_x$ is also continuous on the $x$-axis. Therefore, $f_x$ is continuous everywhere.

Let $x>0$ be given. Then $\big|f(x,y)\big|\leqslant e^{-xy}$. Since the right-hand side function, for given $x>0$, is integrable on $[0,\infty)$, the comparison test implies that $f(x,\cdot)$ is integrable on $[0,\infty)$.

2. Let $x>0$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence with limit 0. W.L.O.G., we can assume that $|h_k|<\dfrac{x}{2}$ since $x>0$. Define

$$g_k(y)=\begin{cases}\dfrac{e^{-yh_k}-1}{h_k}e^{-xy}\dfrac{\sin y}{y} & \text{if } y\neq 0\,,\\ 0 & \text{if } y=0\,.\end{cases}$$

The Mean Value Theorem implies that $\left|\dfrac{e^{-yh_k}-1}{h_k}\right|\leqslant e^{\frac{xy}{2}}|y|$; thus

$$\big|g_k(y)\big|\leqslant e^{-\frac{xy}{2}}\qquad \forall\, y\geqslant 0\,.$$

Since the right-hand side function, for given $x>0$, is integrable on $[0,\infty)$, the Dominated Convergence Theorem implies that

$$\begin{aligned}\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k}&=\lim_{k\to\infty}\int_0^\infty\frac{f(x+h_k,y)-f(x,y)}{h_k}\,dy=\lim_{k\to\infty}\int_0^\infty g_k(y)\,dy\\&=\int_0^\infty\lim_{k\to\infty}g_k(y)\,dy=-\int_0^\infty e^{-xy}\sin y\,dy\end{aligned}$$

Integrating by parts, by the fact $x>0$ we find that

$$\begin{aligned}\int_0^\infty e^{-xy}\sin y\,dy&=-e^{-xy}\cos y\Big|_{y=0}^{y=\infty}-x\int_0^\infty e^{-xy}\cos y\,dy\\&=1-x\Big[e^{-xy}\sin y\Big|_{y=0}^{y=\infty}+x\int_0^\infty e^{-xy}\sin y\,dy\Big]\\&=1-x^2\int_0^\infty e^{-xy}\sin y\,dy\,;\end{aligned}$$

thus we conclude that

$$\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k}=-\frac{1}{1+x^2}$$

for all $x > 0$ and non-zero sequence $\{h_k\}_{k=1}^\infty$ with limit 0. Therefore, for $x > 0$ the limit $\lim\limits_{h\to 0}\dfrac{F(x+h)-F(x)}{h}$ exists (so that $F$ is differentiable on $(0,\infty)$) and

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \frac{1}{1+x^2} \qquad \forall\, x > 0\,.$$

3. By the (generalized version of) Fundamental Theorem of Calculus, for $a, b > 0$ we have

$$F(b) - F(a) = \int_a^b F'(x)\,dx = -\int_a^b \frac{1}{1+x^2}\,dx = \arctan x\Big|_{x=a}^{x=b} = \arctan a - \arctan b\,.$$

Note that for $a > 0$ we have

$$|F(a)| \leqslant \int_0^\infty e^{-ay}\,dy = \frac{e^{-ay}}{-a}\Big|_{y=0}^{y=\infty} = \frac{1}{a}\,;$$

thus $\lim\limits_{a\to\infty} F(a) = 0$ by the Sandwich lemma. Therefore, for $x > 0$,

$$F(x) = \lim_{a\to\infty}\big[F(x) - F(a)\big] = \lim_{a\to\infty}\big(\arctan a - \arctan x\big) = \frac{\pi}{2} - \arctan x\,.$$

Finally, we show that $F(0) = \lim\limits_{x\to 0^+} F(x)$. Let $\varepsilon > 0$ be given. Since

$$\frac{\partial}{\partial y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big) = (e^{-xy}-1)\sin y\,,$$

integrating by parts shows that for all $n > 0$,

$$\begin{aligned}\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy &= \frac{1}{y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\Big|_{y=n}^{y=\infty} \\ &\quad + \int_n^\infty \Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\frac{1}{y^2}\,dy\,.\end{aligned}$$

By the fact that

$$\Big|\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big| \leqslant \frac{x+1}{x^2+1} + 1 \leqslant \frac{5}{2} < 3\,,$$

we have

$$\Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \leqslant \int_n^\infty \frac{3}{y^2}\,dy + \frac{3}{n} = \frac{6}{n}\,.$$

Therefore, for all $n > 0$,

$$\begin{aligned}\big|F(x) - F(0)\big| &= \Big|\int_0^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \Big|\int_0^n (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| + \Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \int_0^n (1-e^{-xy})\,dy + \frac{6}{n} = n + \frac{e^{-nx}-1}{x} + \frac{6}{n}\end{aligned}$$

so that

$$\limsup_{x\to 0^+}\big|F(x) - F(0)\big| \leqslant \frac{6}{n} \qquad \forall\, n > 0\,.$$

Since $n > 0$ is given arbitrarily, we conclude that $\limsup\limits_{x\to 0^+}\big|F(x) - F(0)\big| = 0$ which shows that $\lim\limits_{x\to 0^+} F(x) = F(0)$. As a consequence,

$$\int_0^\infty \frac{\sin x}{x}\,dx = F(0) = \lim_{x\to 0^+} F(x) = \lim_{x\to 0^+}\Big(\frac{\pi}{2} - \arctan x\Big) = \frac{\pi}{2}\,.$$

□