# Exercise Problem Sets 13

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**Problem 1.** Let $\{T_k\}_{k=1}^{\infty} \subseteq \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$ be a sequence of bounded linear maps from $\mathbb{R}^n \to \mathbb{R}^m$. Prove that the following three statements are equivalent:

1. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\{T_k \boldsymbol{x}\}_{k=1}^{\infty}$ converges to $T\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$;
2. $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$;
3. there exists a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that for every compact $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists $N > 0$ such that
$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \varepsilon \quad \text{whenever} \quad \boldsymbol{x} \in K \ \text{ and } \ k \geqslant N\,.$$

*Proof.* "1 ⇒ 3" Let $K$ be a compact set in $\mathbb{R}^n$, and $\varepsilon > 0$ be given. Then there exists $R > 0$ such that $K \subseteq B[0, R]$. By assumption, for each $1 \leqslant i \leqslant n$, there exist $N_i > 0$ such that

$$\|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \frac{\varepsilon}{Rn} \quad \text{whenever} \quad k \geqslant N_i\,.$$

For $\boldsymbol{x} \in \mathbb{R}^n$, write $\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n$. Then if $\boldsymbol{x} \in K$, $\big|x^{(i)}\big| \leqslant R$ for all $1 \leqslant i \leqslant n$. Therefore, if $\boldsymbol{x} \in K$ and $k \geqslant N \equiv \max\{N_1, \cdots, N_n\}$,

$$\begin{aligned}\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} &= \Big\|T_k\Big(\sum_{i=1}^n x^{(i)}\mathbf{e}_i\Big) - T\Big(\sum_{i=1}^n x^{(i)}\mathbf{e}_i\Big)\Big\|_{\mathbb{R}^m} = \Big\|\sum_{i=1}^n x^{(i)}\big(T_k\mathbf{e}_i - T\mathbf{e}_i\big)\Big\|_{\mathbb{R}^m} \\ &\leqslant \sum_{i=1}^n \big|x^{(i)}\big|\|T_k\mathbf{e}_i - T\mathbf{e}_i\|_{\mathbb{R}^m} < \sum_{i=1}^n R\frac{\varepsilon}{Rn} = \varepsilon\,.\end{aligned}$$

"3 ⇒ 2" Let $K = B[0, 1]$ (which is compact), and $\varepsilon > 0$ be given. By assumption there exists $N > 0$ such that

$$\|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{\varepsilon}{3} \quad \text{whenever} \quad \boldsymbol{x} \in B[0, 1] \ \text{ and } \ k \geqslant N\,.$$

If $k, \ell \geqslant N$ and $\boldsymbol{x} \in B[0, 1]$,

$$\|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \|T_k\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} + \|T_\ell\boldsymbol{x} - T\boldsymbol{x}\|_{\mathbb{R}^m} < \frac{2\varepsilon}{3}$$

which shows that

$$\|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = \sup_{\boldsymbol{x}\in B[0,1]} \|T_k\boldsymbol{x} - T_\ell x\|_{\mathbb{R}^m} \leqslant \frac{2\varepsilon}{3} < \varepsilon \qquad \forall\, k, \ell \geqslant N\,.$$

Therefore, $\lim\limits_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{B}(\mathbb{R}^n,\mathbb{R}^m)} = 0$.

"2 ⇒ 1" This part is essentially identical to the proof of Proposition 5.8 in the lecture note (with $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$). □

**Problem 2.** Recall that $\mathcal{M}_{m\times n}$ is the collection of all $m \times n$ real matrices. For a given $A \in \mathcal{M}_{m\times n}$, define a function $f : \mathcal{M}_{n\times m} \to \mathbb{R}$ by

$$f(M) = \operatorname{tr}(AM)\,,$$

where tr is the trace operator which maps a square matrix to the sum of its diagonal entries. Show that $f \in \mathscr{B}(\mathcal{M}_{n\times m}, \mathbb{R})$.

**Hint**: You may need the conclusion in Example 4.29 in the lecture note.

*Proof.* Let $A = [a_{ij}]_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ and $M = [m_{jk}]_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m}$. Then

$$\operatorname{tr}(AM) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} m_{ji}\,.$$

First we show that $f \in \mathscr{L}(\mathcal{M}_{n\times m}, \mathbb{R})$. Let $M = [m_{jk}]_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m}$ and $N = [n_{jk}]_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m}$ be matrices in $\mathcal{M}_{n\times m}$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} f(cM+N) = \operatorname{tr}(A(cM+N)) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(cm_{ji}+n_{ji}) &= c\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}m_{ji} + \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}n_{ji} \\ &= c\operatorname{tr}(AM) + \operatorname{tr}(AN) = cf(M) + f(N)\,. \end{aligned}$$

Let $\|\cdot\| : \mathcal{M}_{n\times m} \to \mathbb{R}$ be defined by

$$\big\|[m_{jk}]_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m}\big\| = \sum_{j=1}^{n}\sum_{k=1}^{m} |m_{jk}|\,.$$

Then $\|\cdot\|$ is a norm on $\mathcal{M}_{n\times m}$, and

$$\sup_{\|M\|=1} |f(M)| = \sup_{\sum_{j=1}^{n}\sum_{k=1}^{m}|m_{jk}|=1} \Big|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}m_{ji}\Big| \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| < \infty\,;$$

thus $f : (\mathcal{M}_{n\times m}, \|\cdot\|) \to (\mathbb{R}, |\cdot|)$ is bounded. Let $|||\cdot|||$ be another norm on $\mathcal{M}_{n\times m}$. Since $\mathcal{M}_{n\times m}$ is finite dimensional vector spaces over $\mathbb{R}$, there exists $c$ and $C$ such that

$$c\|M\| \leqslant |||M||| \leqslant C\|M\| \qquad \forall\, M \in \mathcal{M}_{n\times m}\,.$$

Therefore, $\big\{M \in \mathcal{M}_{n\times m} \,\big|\, |||M||| \leqslant 1\big\} \subseteq \Big\{M \in \mathcal{M}_{n\times m} \,\Big|\, \|M\| \leqslant \dfrac{1}{c}\Big\}$

$$\sup_{|||M|||=1} |f(M)| \leqslant \sup_{\|M\|\leqslant 1/c} |f(M)| = \sup_{\|cM\|\leqslant 1} \frac{1}{c}|f(cM)| \leqslant \frac{1}{c}\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| < \infty\,;$$

thus $f : (\mathcal{M}_{n\times m}, |||\cdot|||) \to \mathbb{R}$ is bounded. □

**Problem 3.** Let $\mathscr{P}([0,1))$ be the collection of all polynomials defined on $[0,1]$, and $\|\cdot\|_\infty$ be the max-norm defined by $\|p\|_\infty = \max\limits_{x\in[0,1]} |p(x)|$.

1. Show that the differential operator $\dfrac{d}{dx} : \mathscr{P}([0,1]) \to \mathscr{P}([0,1])$ is linear.

2. Show that $\frac{d}{dx} : \big(\mathscr{P}([0,1]), \|\cdot\|_\infty\big) \to \big(\mathscr{P}([0,1]), \|\cdot\|_\infty\big)$ is unbounded; that is, show that

$$\sup_{\|p\|_\infty = 1} \|p'\|_\infty = \infty\,.$$

*Proof.* 1. Let $p, q \in \mathscr{P}([0,1])$ and $c \in \mathbb{R}$. Then by the rule of differentiation,

$$\frac{d}{dx}(cp+q)(x) = cp'(x) + q'(x) = c\frac{d}{dx}p(x) + \frac{d}{dx}q(x)\,;$$

thus $\frac{d}{dx} : \mathscr{P}([0,1]) \to \mathscr{P}([0,1])$ is linear.

2. Consider $p_n(x) = x^n$. Then $\|p_n\|_\infty = \max\limits_{x\in[0,1]} x^n = 1$ for all $n \in \mathbb{N}$; however,

$$\|p_n'\|_\infty = \max_{x\in[0,1]} nx^{n-1} = n \qquad n \in \mathbb{N}\,;$$

thus $\sup\limits_{\|p\|_\infty=1} \|p'\|_\infty = \infty$. □

**Problem 4.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $T \in \mathscr{B}(X,Y)$. Show that for all $\boldsymbol{x} \in X$ and $r > 0$,

$$\sup_{\boldsymbol{x}' \in B(\boldsymbol{x}, r)} \|T\boldsymbol{x}'\|_Y \geqslant r\|T\|_{\mathscr{B}(X,Y)}\,.$$

**Hint**: Prove and make use of the inequality $\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y, \|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\} \geqslant \|T\boldsymbol{\xi}\|_Y$ for all $\boldsymbol{\xi} \in Y$.

*Proof.* Let $\boldsymbol{x} \in X$ and $r > 0$ be given. Then for all $\boldsymbol{\xi} \in B(0, r)$,

$$\begin{aligned} &\max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y, \|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\} \\ &\quad\geqslant \frac{1}{2}\Big[\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y + \|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\Big] \geqslant \frac{1}{2}\|T(\boldsymbol{x}+\boldsymbol{\xi}) - T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y = \|T\boldsymbol{\xi}\|_Y\,. \end{aligned}$$

Therefore,

$$\sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)} \max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y, \|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\} \geqslant \sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)} \|T\boldsymbol{\xi}\|_Y = r\|T\|_{\mathscr{B}(X,Y)}\,,$$

and the desired inequality follows from the fact that

$$\sup_{\boldsymbol{x}'\in B(\boldsymbol{x},r)} \|T\boldsymbol{x}'\|_Y = \sup_{\boldsymbol{\xi}\in B(\boldsymbol{0},r)} \max\big\{\|T(\boldsymbol{x}+\boldsymbol{\xi})\|_Y, \|T(\boldsymbol{x}-\boldsymbol{\xi})\|_Y\big\}\,.$$ □

**Problem 5.** Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ be a normed space, and $\mathscr{F} \subseteq \mathscr{B}(X,Y)$ be a family of bounded linear maps from $X$ to $Y$. Show that if $\sup\limits_{T\in\mathscr{F}} \|T\boldsymbol{x}\|_Y < \infty$ for all $x \in X$, then

$$\sup_{T\in\mathscr{F}} \|T\|_{\mathscr{B}(X,Y)} < \infty\,.$$

**Hint**: Suppose the contrary that there exists $\{T_n\}_{n=1}^\infty \subseteq \mathscr{F}$ such that $\|T_n\|_{\mathscr{B}(X,Y)} \geqslant 4^n$. Using Problem 4 to choose a sequence $\{\boldsymbol{x}_n\}_{n=0}^\infty$, where $\boldsymbol{x}_0 = \boldsymbol{0}$, such that

$$\boldsymbol{x}_n \in B(\boldsymbol{x}_{n-1}, 3^{-n}) \qquad \text{and} \qquad \|T_n\boldsymbol{x}_n\|_Y \geqslant \frac{2}{3}\cdot 3^{-n}\|T_n\|_{\mathscr{B}(X,Y)}\,.$$

Show that $\{\boldsymbol{x}_n\}_{n=1}^\infty$ converges to some point $\boldsymbol{x} \in X$ but $\{T_n\boldsymbol{x}\}_{n=1}^\infty$ is not bounded in $Y$.

**Remark**: The conclusion above is called the Uniform Boundedness Principle (or the Banach-Steinhaus Theorem). This is one of the fundamental results in functional analysis.

*Proof.* Suppose the contrary that $\sup_{T\in\mathscr{F}} \|T\|_{\mathscr{B}(X,Y)} = \infty$. Then there exists $\{T_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ such that

$$\|T_n\|_{\mathscr{B}(X,Y)} \geqslant 4^n \qquad \forall\, n \in \mathbb{N}\,.$$

Let $\boldsymbol{x}_0 = \mathbf{0}$. Define $r_n = 3^{-n}$ and $\{\boldsymbol{x}_n\}_{n=1}^{\infty} \subseteq X$ so that

$$\boldsymbol{x}_n \in B(\boldsymbol{x}_{n-1}, r_n) \qquad \text{and} \qquad \|T_n\boldsymbol{x}_n\|_Y \geqslant \frac{2}{3} r_n \|T_n\|_{\mathscr{B}(X,Y)}\,.$$

We note that such $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ exists because of Problem 4. For $m > n$,

$$\begin{aligned}\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_X &\leqslant \|\boldsymbol{x}_n - \boldsymbol{x}_{n+1}\|_X + \|\boldsymbol{x}_{n+1} - \boldsymbol{x}_{n+1}\|_X + \cdots + \|\boldsymbol{x}_{m-1} - \boldsymbol{x}_m\|_X \\ &\leqslant 3^{-(n+1)} + 3^{-(n+2)} + \cdots + 3^{-m} \leqslant 3^{-(n+1)}\Big(1 + \frac{1}{3} + \cdots\Big) \leqslant \frac{1}{2}\cdot 3^{-n}\,;\end{aligned}$$

thus $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X, \|\cdot\|_X)$ is complete, $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ converges to some point $\boldsymbol{x} \in X$, and $\|\boldsymbol{x} - \boldsymbol{x}_n\|_X \leqslant \frac{1}{2}\cdot 3^{-n}$. Therefore,

$$\begin{aligned}\|T_n\boldsymbol{x}\|_Y &\geqslant \|T_n\boldsymbol{x}_n\|_Y - \|T_n(\boldsymbol{x} - \boldsymbol{x}_n)\|_Y \geqslant \frac{2}{3} r_n \|T_n\|_{\mathscr{B}(X,Y)} - \|T_n\|_{\mathscr{B}(X,Y)} \|\boldsymbol{x} - \boldsymbol{x}_n\|_X \\ &\geqslant \Big(\frac{2}{3} - \frac{1}{2}\Big)\|T_n\|_{\mathscr{B}(X,Y)} 3^{-n} = \frac{1}{6}\|T_n\|_{\mathscr{B}(X,Y)} 3^{-n} \geqslant \frac{1}{6}\cdot\Big(\frac{4}{3}\Big)^n\end{aligned}$$

so that $\sup_{n\in\mathbb{N}} \|T_n\boldsymbol{x}\|_Y = \infty$, a contradiction. ◻