

Exercise Problem Sets 12

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此次習題是與均勻連續相關的習題，僅問題 5、8、11 列入可能期中考的範圍。習題解答下星期二公布。

Problem 1. Check if the following functions on uniformly continuous.

1. $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \sin \log x$.
2. $f : (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) = x \sin \frac{1}{x}$.
3. $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$.
4. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \cos(x^2)$.
5. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \cos^3 x$.
6. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x \sin x$.

Problem 2. 1. Find all positive numbers a and b such that the function $f(x) = \frac{\sin(x^a)}{1+x^b}$ is uniformly continuous on $[0, \infty)$.

2. Find all positive numbers a and b such that the function $f(x, y) = |x|^a |y|^b$ is uniformly continuous on \mathbb{R}^2 .

Problem 3. Show that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x, y) = \frac{\sqrt{x}}{1+x^2 y^2}$ is uniformly continuous on its domain.

Proof. Let $\varepsilon > 0$ be given. Choose $N > 0$ so that $\frac{8}{4 + \varepsilon^2 N^2} < \varepsilon$. Then

1. if $(x, y) \in [0, \frac{\varepsilon^2}{4}] \times [-N, N]^c$, we have $|f(x, y)| \leq \sqrt{x} < \frac{\varepsilon}{2}$.
2. if $(x, y) \in [\frac{\varepsilon^2}{4}, 1] \times [-N, N]^c$, we have $|f(x, y)| \leq \frac{1}{1 + \frac{\varepsilon^2}{4} N^2} = \frac{4}{4 + \varepsilon^2 N^2} < \frac{\varepsilon}{2}$.

Therefore,

$$|f(x, y)| \leq \frac{\varepsilon}{2} \quad \forall (x, y) \in [0, 1] \times [-N, N]^c.$$

Since $[0, 1] \times [-2N, 2N]$ is compact, the continuity of f implies that f is uniformly continuous on $[0, 1] \times [0, 2N]$; thus there exists $\delta_1 > 0$ such that

$$|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon \quad \forall |(x_1, y_1) - (x_2, y_2)| < \delta_1 \text{ and } x_1, x_2 \in [0, 1], y_1, y_2 \in [-2N, 2N].$$

Define $\delta = \min\{\delta_1, N\}$. If $(x_1, y_1), (x_2, y_2) \in [0, 1] \times \mathbb{R}$ and $|(x_1, y_1) - (x_2, y_2)| < \delta$, then either $(x_1, y_1), (x_2, y_2)$ belongs to $[0, 1] \times [-2N, 2N]$ or $(x_1, y_1), (x_2, y_2)$ belongs to $[0, 1] \times [-N, N]^c$. Therefore,

$$|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon \quad \forall (x_1, y_1), (x_2, y_2) \in [0, 1] \times \mathbb{R} \text{ and } |(x_1, y_1) - (x_2, y_2)| < \delta. \quad \square$$

Problem 4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuous, and $\lim_{|x| \rightarrow \infty} f(x) = b$ exists for some $b \in \mathbb{R}^m$. Show that f is uniformly continuous on \mathbb{R}^n .

Proof. Let $\varepsilon > 0$ be given. By the fact that $\lim_{|x| \rightarrow \infty} f(x) = b$, there exists $M > 0$ such that

$$\|f(x) - b\|_{\mathbb{R}^m} < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x\|_{\mathbb{R}^n} \geq M.$$

By the Heine-Borel Theorem, $B[0, M + 1]$ is compact; thus f is uniformly continuous on $B[0, M + 1]$ and there exists $\delta \in (0, \frac{1}{2})$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta \text{ and } x, y \in B[0, M + 1]. \quad (\star)$$

Therefore, for $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| < \delta$,

1. if $x, y \in B[0, M + 1]$, then (\star) implies that

$$\|f(x) - f(y)\|_{\mathbb{R}^m} < \varepsilon.$$

2. if $x \notin B[0, M + 1]$ or $y \notin B[0, M + 1]$, then $x, y \in B[0, M]^c$ which implies that

$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leq \|f(x)\|_{\mathbb{R}^m} + \|f(y)\|_{\mathbb{R}^m} < \varepsilon. \quad \square$$

Problem 5. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is uniformly continuous. Show that there exists $a > 0$ and $b > 0$ such that $\|f(x)\|_{\mathbb{R}^m} \leq a\|x\|_{\mathbb{R}^n} + b$.

Proof. Since f is uniformly continuous on \mathbb{R}^n , there exists $\delta > 0$ such that

$$\|f(x) - f(y)\|_{\mathbb{R}^m} < 1 \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta.$$

For a given $x \in \mathbb{R}^n$, let $N \in \mathbb{N}$ such that $\frac{\|x\|_{\mathbb{R}^n}}{\delta} < N \leq \frac{\|x\|_{\mathbb{R}^n}}{\delta} + 1$. For each $k \in \mathbb{N}$, define points x_k by $x_k \equiv \frac{kx}{N}$. Then $\{x_k\}_{k=0}^{\infty}$ satisfies that

$$\|x_k - x_{k-1}\|_{\mathbb{R}^n} = \frac{\|x\|_{\mathbb{R}^n}}{N} < \delta \quad \forall k \in \mathbb{N}$$

which further implies that

$$\|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} < 1 \quad \forall k \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} \|f(x)\|_{\mathbb{R}^m} &\leq \|f(x) - f(0)\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \leq \sum_{k=1}^N \|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \\ &\leq N + \|f(0)\|_{\mathbb{R}^m} \leq \frac{1}{\delta} \|x\|_{\mathbb{R}^n} + \|f(0)\|_{\mathbb{R}^m} + 1; \end{aligned}$$

thus $a = \frac{1}{\delta}$ and $b = \|f(0)\|_{\mathbb{R}^m} + 1$ verify the inequality $\|f(x)\|_{\mathbb{R}^m} \leq a\|x\|_{\mathbb{R}^n} + b$. \square

Problem 6. Let $f(x) = \frac{q(x)}{p(x)}$ be a rational function defined on \mathbb{R} , where p and q are two polynomials. Show that f is uniformly continuous on \mathbb{R} if and only if the degree of q is not more than the degree of p plus 1.

Proof. Note that if f is defined on \mathbb{R} , then $p(x) \neq 0$ for all $x \in \mathbb{R}$. By Problem 5, there exist $a, b > 0$ such that

$$\left| \frac{q(x)}{p(x)} \right| \leq a|x| + b \quad \forall x \in \mathbb{R}.$$

Therefore, $|q(x)| \leq |p(x)|(a|x| + b)$ for all $x \in \mathbb{R}$, and this inequality above can be true if and only if the degree of q is not more than the degree of p plus 1. \square

Problem 7. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous periodic function; that is, there exists $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on \mathbb{R} .

Proof. Let $p > 0$ be such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$, and $\varepsilon > 0$ be given. Since f is uniformly continuous on $[-p, p]$, there exists $\delta \in (0, p)$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta \text{ and } x, y \in [-p, p].$$

Therefore, if $|x - y| < \delta$, we must have $x, y \in [kp - p, kp + p]$ for some $k \in \mathbb{Z}$ so that $x - kp, y - kp \in [-p, p]$ which, together with the fact that $|(x - kp) - (y - kp)| = |x - y| < \delta$, implies that

$$|f(x) - f(y)| = |f(x - kp) - f(y - kp)| < \varepsilon. \quad \square$$

Problem 8. Let $(a, b) \subseteq \mathbb{R}$ be an open interval, and $f : (a, b) \rightarrow \mathbb{R}^m$ be a function. Show that the following three statements are equivalent.

1. f is uniformly continuous on (a, b) .
2. f is continuous on (a, b) , and both limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exist.
3. For all $\varepsilon > 0$, there exists $N > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\left| \frac{f(x) - f(y)}{x - y} \right| > N$ and $x, y \in (a, b)$, $x \neq y$.

Proof. First we note that 1 and 2 are equivalent since

1. if f is uniformly continuous on (a, b) , then there is a unique continuous extension g of f on $[a, b]$; thus $\lim_{x \rightarrow a^+} g(x) = g(a)$ and $\lim_{x \rightarrow b^-} g(x) = g(b)$ exists, and 2 holds since $\lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} g(x) = \lim_{x \rightarrow b^-} f(x)$.
2. if $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exists, we define $g : [a, b] \rightarrow \mathbb{R}$ by $g(x) = f(x)$ for $x \in (a, b)$ and $g(a), g(b)$ are respectively the limit of f at a, b . Then g is continuous on $[a, b]$; thus the compactness of $[a, b]$ shows that g is uniformly continuous on $[a, b]$. In particular, g is uniformly continuous on (a, b) which is the same as saying that f is uniformly continuous on (a, b) .

Next we prove that 1 and 3 are equivalent.

“1 \Rightarrow 3” Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in (a, b)$ such that

$$x_n \neq y_n, \quad |f(x_n) - f(y_n)| \geq \varepsilon \quad \text{but} \quad \left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right| > n \quad \forall n \in \mathbb{N}.$$

By the Bolzano-Weierstrass Theorem/Property, there exist convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ and $\{y_{n_j}\}_{j=1}^{\infty}$ with limit x and y . Since $x_n, y_n \in (a, b)$ for all $n \in \mathbb{N}$, we must have $x, y \in [a, b]$. If $x = y$, then $|x_n - y_n| \rightarrow 0$ as $n \rightarrow \infty$; thus the uniform continuity of f on (a, b) implies that $|f(x_n) - f(y_n)| \rightarrow 0$ as $n \rightarrow \infty$ which contradicts to the fact that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Therefore, $x \neq y$ which further shows that the limit

$$\lim_{n \rightarrow \infty} \left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right|$$

exists since the limit $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ both exist and $\lim_{n \rightarrow \infty} (x_n - y_n) = x - y \neq 0$.

This is a contradiction to that $\left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right| > n$ for all $n \in \mathbb{N}$.

“3 \Rightarrow 1” Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in (a, b)$ satisfying $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. For this $\varepsilon > 0$, by assumption there exists $N > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad \left| \frac{f(x) - f(y)}{x - y} \right| > N \quad \text{and} \quad x, y \in (a, b), x \neq y.$$

Since $|f(x_n) - f(y_n)| \geq \varepsilon$, we must have $x_n \neq y_n$; thus the fact that $x_n, y_n \in (a, b)$ implies that

$$\left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right| \leq N \quad \forall n \in \mathbb{N}.$$

This contradicts to the fact that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. □

Problem 9. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is **Hölder continuous with exponent** α ; that is, there exist $M > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha \quad \forall x, y \in [a, b].$$

Show that f is uniformly continuous on $[a, b]$. Show that $f : [0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\frac{1}{2}$.

Proof. Let $\varepsilon > 0$ be given. Define $\delta = M^{-\frac{1}{\alpha}} \varepsilon^{\frac{1}{\alpha}}$. Then $\delta > 0$. Moreover, if $|x - y| < \delta$ and $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|^\alpha < M\delta^\alpha = \varepsilon.$$

Therefore, f is uniformly continuous on $[a, b]$.

Next we show that $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\frac{1}{2}$. Note that if $x, y \geq 0$ and $x \neq y$,

$$\frac{|\sqrt{x} - \sqrt{y}|}{|x - y|^{\frac{1}{2}}} = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|x - y|^{\frac{1}{2}}|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|^{\frac{1}{2}}}{|\sqrt{x} + \sqrt{y}|} \leq \frac{\sqrt{x} + \sqrt{y}}{|\sqrt{x} + \sqrt{y}|} \leq 1;$$

thus

$$|\sqrt{x} - \sqrt{y}| \leq |x - y|^{\frac{1}{2}} \quad \forall x, y \geq 0 \text{ and } x \neq y.$$

which implies that $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0, \infty)$. \square

Problem 10. A function $f : A \times B \rightarrow \mathbb{R}^m$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^p$, is said to be separately continuous if for each $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. f is said to be continuous on A uniformly with respect to B if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \|f(x, y) - f(x_0, y)\|_2 < \varepsilon \quad \text{whenever} \quad \|x - x_0\|_2 < \delta \text{ and } y \in B.$$

Show that if f is separately continuous and is continuous on A uniformly with respect to B , then f is continuous on $A \times B$.

Proof. Let $\varepsilon > 0$, and $(a, b) \in A \times B$ be given. By assumption there exists $\delta_1 > 0$ such that

$$\|f(x, y) - f(a, y)\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x - a\|_2 < \delta_1 \text{ and } x \in A, y \in B.$$

Since f is separately continuous, there exists $\delta_2 > 0$ such that

$$\|f(a, y) - f(a, b)\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|y - b\|_2 < \delta_2 \text{ and } y \in B.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then if $\|(x, y) - (a, b)\|_2 < \delta$, we must have $\|x - a\|_2 < \delta_1$ and $\|y - b\|_2 < \delta_2$ so that

$$\begin{aligned} \|f(x, y) - f(a, b)\|_2 &= \|f(x, y) - f(a, y) + f(a, y) - f(a, b)\|_2 \\ &\leq \|f(x, y) - f(a, y)\|_2 + \|f(a, y) - f(a, b)\|_2 < \varepsilon \end{aligned}$$

which shows that f is continuous at (a, b) . \square

Problem 11. Let (M, d) be a metric space, $A \subseteq M$, and $f, g : A \rightarrow \mathbb{R}$ be uniformly continuous on A . Show that if f and g are bounded, then fg is uniformly continuous on A . Does the conclusion still hold if f or g is not bounded?

Proof. Let $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ be sequences in A satisfying that $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Suppose that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in A$. Then

$$\begin{aligned} |f(x_n)g(x_n) - f(y_n)g(y_n)| &= |f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)| \\ &\leq |f(x_n)||g(x_n) - g(y_n)| + |g(y_n)||f(x_n) - f(y_n)| \\ &\leq M(|f(x_n) - f(y_n)| + |g(x_n) - g(y_n)|); \end{aligned}$$

thus the uniform continuity of f and g , together with the Sandich Lemma, implies that

$$\lim_{n \rightarrow \infty} |f(x_n)g(x_n) - f(y_n)g(y_n)| = 0.$$

Therefore, fg is uniformly continuous on A .

When the boundedness is removed from the condition, the product of f and g might not be uniformly continuous. For example, $f(x) = g(x) = x$ are continuous on \mathbb{R} , but $(fg)(x) = x^2$ is not uniformly continuous on \mathbb{R} (from an example in class). \square