Exercise Problem Sets 10

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本週習題請同學除了以下三題之外，也持續看一下習題九還沒講解的習題。

Problem 1 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Every open set in a metric space is a countable union of closed sets.
2. Let $A \subseteq \mathbb{R}$ be bounded from above, then $\sup A \in A'$.
3. An infinite union of distinct closed sets cannot be closed.
4. An interior point of a subset A of a metric space (M, d) is an accumulation point of that set.
5. Let (M, d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.
6. There exists a metric space in which some unbounded Cauchy sequence exists.
7. Every metric defined in $\mathbb{R}^n$ is induced from some "norm" in $\mathbb{R}^n$.
8. There exists a non-zero dimensional normed vector space in which some compact non-zero dimensional linear subspace exists.
9. There exists a set $A \subseteq (0, 1]$ which is compact in $(0, 1]$ (in the sense of subspace topology), but A is not compact in $\mathbb{R}$.
10. Let $A \subseteq \mathbb{R}^n$ be a non-empty set. Then a subset B of A is compact in A if and only if B is closed and bounded in A.

Solution. 1. **True**. We note that the statement above is equivalent to that "every closed set in a metric space is a countable intersection of open sets". To see that this equivalent statement is true, we let F be a closed set. For each $n \in \mathbb{N}$, define

$$U_n = \bigcup_{x \in F} B\big(x, \frac{1}{n}\big)\,.$$

Then $F \subseteq U_n$ $\big($since each point $x \in F$ belongs to the ball $B\big(x, \frac{1}{n}\big)\big)$. Moreover, U_n is open since it is the union of open sets.

Claim: $F = \bigcap_{n=1}^{\infty} U_n$.

Proof of claim: Since $F \subseteq U_n$ for all $n \in \mathbb{N}$, $F \subseteq \bigcap_{n=1}^{\infty} U_n$; thus it suffices to shows that $F \supseteq \bigcap_{n=1}^{\infty} U_n$ or equivalently, $F^\complement \subseteq \bigcup_{n=1}^{\infty} U_n^\complement$. To see the inclusion, we let $x \in F^\complement$ and use the closedness of F to find an $n_0 \in \mathbb{N}$ such that $B(x, \frac{1}{n_0}) \subseteq F^\complement$. This implies that $d(x, y) \geqslant \frac{1}{n_0}$ for all $y \in F$; thus $x \notin U_{n_0}$. Therefore, $x \in U_{n_0}^\complement$ so that $x \in \bigcup_{n=1}^{\infty} U_n^\complement$. □

2. **False**. Let A be a collection of single point $\{a\}$. Then A is bounded from above and $\sup A = a$ but $A' = \varnothing$.

3. **False**. Consider the union of the family of closed sets $\big\{[3n-1, 3n+1] \,\big|\, n \in \mathbb{N}\big\}$. We note that for $n \neq m$ the two sets $[3n-1, 3n+1] \cap [3m-1, 3m+1] = \varnothing$ so that this family is a collection of distinct set and $\bigcup\limits_{n=1}^{\infty} [3n-1, 3n+1]$ is closed.

4. **False**. Every point x in a discrete metric is the only point in the set $B(x, 1)$ so that $x \notin B(x, 1)'$.

5. **False**. A counter-example can be found in 5 of Problem 9 in Exercise 8.

6. **False**. By Proposition 2.58 in the lecture note, every Cauchy sequence is bounded.

7. **False**. The discrete metric d_0 on $\mathbb{R}^n$ cannot be induced by a norm since every set in $(\mathbb{R}^n, d_0)$ is bounded but $\mathbb{R}^n$ is unbounded in $(\mathbb{R}^n, \|\cdot\|)$ for any norms $\|\cdot\|$ on $\mathbb{R}^n$.

8. **False**. Note that any non-zero dimensional linear subspace of a normed space is unbounded; thus any non-zero dimensional linear subspace cannot be compact since a compact set must be bounded.

9. **False**. By Theorem 3.77 in the lecture note, A is compact in $(0, 1]$ if and only if A is compact in $\mathbb{R}$.

10. **False**. By Theorem 3.42 in the lecture note, it is true that B is compact in A then B is closed and bounded in A; however, the reverse statement if not true. For example, if $A = B = (0, 1)$, then B is closed and bounded in A but B is not compact in $\mathbb{R}$. □

Problem 2. Let (M, d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.

2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.

3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.

4. $\text{cl}(\text{int}(A)) = A$.

5. $\partial(\text{cl}(A)) = \partial A$.

6. If A is open, then $\partial A \subseteq M \backslash A$.

7. If A is open, then $A = \text{cl}(A) \backslash \partial A$. How about if A is not open?

Solution. 1. **True**. First we note that $\mathring{A} \subseteq A$ and $\mathring{A} \cap \partial A = \varnothing$. Therefore,

$$\mathring{A} \subseteq A \backslash \partial A\,.$$

On the other hand, if $x \in A \backslash \partial A$, by the fact that $\partial A = \bar{A} \cap \overline{A^{\complement}}$, we find that x is not a limit point of $A^{\complement}$; thus there exists $r > 0$ such that $B(x,r) \subseteq (A^{\complement})^{\complement} = A$. This Remark 3.3 in the lecture note implies that $x \in \mathring{A}$ so that $A \backslash \partial A \subseteq \mathring{A}$.

2. **True**. Note that $x \notin \mathring{B}$ if and only if there exists $\{x_n\}_{n=1}^{\infty} \subseteq B^{\complement}$ such that $\lim\limits_{n \to \infty} x_n = x$. Therefore,

$$x \in \bar{A} \Leftrightarrow \big(\exists \{x_n\}_{n=1}^{\infty} \subseteq A\big)\big(\lim_{n \to \infty} x_n = x\big) \Leftrightarrow \big(\exists \{x_n\}_{n=1}^{\infty} \subseteq (M \backslash A)^{\complement}\big)\big(\lim_{n \to \infty} x_n = x\big)$$
$$\Leftrightarrow x \notin \operatorname{int}(M \backslash A) \Leftrightarrow x \in M \backslash \operatorname{int}(M \backslash A)\,.$$

3. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\operatorname{cl}(A) = [0,1]$ and $\operatorname{int}(A) = \varnothing$ so that $\operatorname{int}(\operatorname{cl}(A)) = (0,1) \neq \operatorname{int}(A)$.

4. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\operatorname{int}(A) = \varnothing$ so that $\operatorname{cl}(\operatorname{int}(A)) = \varnothing \neq A$.

5. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\bar{A} = [0,1]$ so that $\partial \bar{A} = \{0,1\} \neq \partial A$.

6. **True**. If A is open, then every point $x \in A$ is an interior point so that $x \notin \partial A$ (if $x \in \partial A$, then there exists $\{x_n\}_{n=1}^{\infty} \subseteq A^{\complement}$ such that $\lim\limits_{n \to \infty} x_n = x$ so that $x \notin \mathring{A}$).

7. **True**. By Proposition 3.13 in the lecture note, $\partial A = \bar{A} \backslash \mathring{A}$; thus the fact that $\mathring{A} \subseteq \bar{A}$ shows that $\bar{A} = \mathring{A} \cup \partial A$. Since $\partial A \cap \mathring{A} = \varnothing$, we find that $A = \bar{A} \backslash \partial A$.

   If A is not open, the statement is false. For example, consider $A = [0,1]$ in $(\mathbb{R}, |\cdot|)$. Then A is not open and $\bar{A} = [0,1]$ and $\partial A = \{0,1\}$ so that $\bar{A} \backslash \partial A = (0,1) \neq A$. $\square$

Problem 3. Use whatever methods you know to find the following limits:

1. $\lim\limits_{x \to 0^+} (1 + \sin 2x)^{\frac{1}{x}}$;
2. $\lim\limits_{x \to -\infty} \big(\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}\big)$;
3. $\lim\limits_{x \to 1} (2 - x)^{\sec \frac{\pi x}{2}}$;
4. $\lim\limits_{x \to \infty} x\Big(\dfrac{\pi}{2} - \sin^{-1} \dfrac{x}{\sqrt{x^2 + 1}}\Big)$;
5. $\lim\limits_{x \to \infty} x\Big(e^{-1} - \big(\dfrac{x}{x+1}\big)^x\Big)$;
6. $\lim\limits_{x \to \infty} \Big(\dfrac{a^x - 1}{x(a-1)}\Big)^{\frac{1}{x}}$, where $a > 0$ and $a \neq 1$.