Exercise Problem Sets 7

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Problem 1. Let $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two sequences of real numbers, and $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ converges if $\sum\limits_{n=1}^{\infty} a_n$ converges.

Proof. First we note that if $n > m$,

$$\begin{aligned} |x_n - x_m| &= \big|x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m\big| \\ &\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &\leqslant a_{n-1} + a_{n-2} + \cdots + a_m = \sum_{k=m}^{n-1} a_k \,. \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\sum\limits_{k=1}^{\infty} a_k$ converges, the Cauchy criterion implies that there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+p} a_k\Big| = \big|a_n + a_{n+1} + \cdots + a_{n+p}\big| < \varepsilon \quad \text{whenever} \quad n \geqslant N \text{ and } p \geqslant 0\,.$$

Therefore, if $n > m \geqslant N$, by the fact $a_k > 0$ for all $k \in \mathbb{N}$, we have

$$|x_n - x_m| \leqslant \sum_{k=m}^{n-1} a_k < \varepsilon\,.$$

This implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$ converges.
□

Problem 2. Let $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ ba a sequence. A series $\sum\limits_{k=1}^{\infty} b_k$ is said to be a rearrangement of the series $\sum\limits_{k=1}^{\infty} a_k$ if there exists a rearrangement π of $\mathbb{N}$; that is, $\pi : \mathbb{N} \to \mathbb{N}$ is bijective, such that $b_k = a_{\pi(k)}$.

1. Show that if $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of the series $\sum\limits_{k=1}^{\infty} a_k$ converges and has the value $\sum\limits_{k=1}^{\infty} a_k$.
2. Show that if $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent, then for each $r \in \mathbb{R}$, there exists a rearrangement $\sum\limits_{k=1}^{\infty} a_{\pi(k)}$ of the series $\sum\limits_{k=1}^{\infty} a_k$ such that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$.

Proof. 1. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is an absolutely convergent series with limit a, and $\pi : \mathbb{N} \to \mathbb{N}$ is a rearrangement of $\mathbb{N}$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$\Big|\sum_{k=1}^{n} a_k - a\Big| < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{k=n+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N\,.$$

Choose $K > 0$ such that $\pi(n) > N$ if $n \geqslant K$. In fact, $K = \max\big\{\pi^{-1}(1), \cdots, \pi^{-1}(N)\big\} + 1$ suffices the purpose. Then $K \geqslant N$ and if $n \geqslant K$, $\pi\big(\{1, 2, \cdots, n\}\big) \supseteq \{1, 2, \cdots, N\}$. Therefore, if $n \geqslant K$,

$$\Big|\sum_{k=1}^{n} a_{\pi(k)} - a\Big| \leqslant \Big|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{N} a_k\Big| + \Big|\sum_{k=1}^{N} a_k - a\Big| \leqslant \sum_{k=N+1}^{\infty} |a_k| + \frac{\varepsilon}{2} < \varepsilon$$

which implies that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = a$.

2. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent. Let $\{a_{k_j}\}_{j=1}^{\infty}$ denote the subsequence of $\{a_k\}_{k=1}^{\infty}$ so that $a_{k_j} \geqslant 0$ for all $j \in \mathbb{N}$ and $a_k < 0$ if $k \in \mathbb{N}\backslash\{k_1, k_2, \cdots\}$. In other words, $\{a_{p_j}\}_{j=1}^{\infty}$ is the maximal subsequence of $\{a_k\}_{k=1}^{\infty}$ with non-negative terms. Let $\{a_{n_j}\}_{j=1}^{\infty}$ be the maximal subsequence of $\{a_k\}_{k=1}^{\infty}$ with negative terms. Then

$$\sum_{j=1}^{\infty} a_{p_j} = \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} a_{n_j} = -\infty\,.$$

Let $r \in \mathbb{R}$ be given, and use the notation $\sum\limits_{j=1}^{0}$ to denote summing nothing. Define $k_0 = 0$. Choose $k_1 \in \mathbb{N}$ be the unique natural number so that $\sum\limits_{j=1}^{k_1-1} a_{p_j} < r$ but $\sum\limits_{j=1}^{k_1} a_{p_j} > r$. Since $\sum\limits_{j=1}^{\infty} a_{n_j} = -\infty$, there exists a unique $k_2 \in \mathbb{N}$ such that $\sum\limits_{j=1}^{k_1} a_{p_j} + \sum\limits_{j=1}^{k_2-1} a_{n_j} > r$ but $\sum\limits_{j=1}^{k_1} a_{p_j} + \sum\limits_{j=1}^{k_2} a_{n_j} < r$. We continue this process, and obtain a sequence $\{k_\ell\}_{\ell=0}^{\infty}$ such that for each $\ell \in \mathbb{N}$,

(a) $\sum\limits_{j=1}^{k_{2\ell-1}-1} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell-2}} a_{n_j} < r.$

(b) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell-2}} a_{n_j} > r.$

(c) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell}-1} a_{n_j} > r.$

(d) $\sum\limits_{j=1}^{k_{2\ell-1}} a_{p_j} + \sum\limits_{j=1}^{k_{2\ell}} a_{n_j} < r.$

We then obtain a permutation of $\{a_n\}_{n=1}^{\infty}$:

$$\underbrace{a_{p_1}, \cdots, a_{p_{k_1}}}_{k_1 \text{ ``}\geqslant 0\text{'' terms}}, \underbrace{a_{n_1}, \cdots, a_{n_{k_2}}}_{k_2 \text{ ``}< 0\text{'' terms}}, \underbrace{a_{p_{k_1+1}}, \cdots, a_{p_{k_3}}}_{k_3 \text{ ``}\geqslant 0\text{'' terms}}, \underbrace{a_{n_{k_2+1}}, \cdots, a_{n_{k_4}}}_{k_4 \text{ ``}< 0\text{'' terms}}, \cdots.$$

Denote the permutation above by $\{a_{\pi(n)}\}_{n=1}^{\infty}$; that is, $\pi(1) = p_1$, $\cdots$, $\pi(k_1) = p_{k_1}$, $\pi(k_1+1) = n_1$, $\cdots$, $\pi(k_1 + k_2) = n_{k_2}$, and so on. Next we show that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$.

Let $\varepsilon > 0$ be given, and define $S_n = \sum\limits_{k=1}^{n} a_{\pi(k)}$. Since $\sum\limits_{n=1}^{\infty} a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$; thus there exists $N > 0$ such that $|a_n| < \varepsilon$ for all $n \geqslant N$. By the construction of $\{k_\ell\}_{\ell=1}^{\infty}$,

$$|S_n - S_{n-1}| = |a_{\pi(n)}| < \varepsilon \quad \text{whenever} \quad n \geqslant k_1 + k_2 + \cdots + k_N\,.$$

This implies that $S_n \in (r - \varepsilon, r + \varepsilon)$ whenever $n \geqslant k_1 + k_2 + \cdots + k_N$. Therefore,

$$\Big|\sum_{k=1}^{n} a_{\pi(k)} - r\Big| < \varepsilon \quad \text{whenever} \quad n \geqslant k_1 + k_2 + \cdots + k_N$$

which shows that $\sum\limits_{k=1}^{\infty} a_{\pi(k)} = r$. $\square$

Alternative proof of 1. We first establish the following

Claim: If $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation, then $\sum\limits_{n=1}^{\infty} a_{\pi(n)} = \sum\limits_{n=1}^{\infty} a_n$.

To see the claim, let $\{a_n\}_{n=1}^{\infty}$ be non-negative sequence and $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. By the fact that $a_n \geqslant 0$ for all $n \geqslant \mathbb{N}$, we find that for all $N \in \mathbb{N}$,

$$S_N \equiv \sum_{n=1}^{N} a_{\pi(n)} \leqslant \sum_{n=1}^{\infty} a_n .$$

Since $\{S_N\}_{N=1}^{\infty}$ is an increasing sequence, $\lim\limits_{N\to\infty} S_N$ either exists or diverges to ∞. In either cases,

$$\sum_{n=1}^{\infty} a_{\pi(n)} = \lim_{N\to\infty} S_N \leqslant \sum_{n=1}^{\infty} a_n . \qquad (\diamond)$$

On the other hand, we also note that $\sum\limits_{n=1}^{\infty} a_n$ is a rearrangement of $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$. In fact, if $b_n = a_{\pi(n)}$, then $a_n = b_{\pi^{-1}(n)}$ so that $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} b_{\pi^{-1}(n)}$. Therefore, $(\diamond)$ implies that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_{\pi^{-1}(n)} \leqslant \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{\pi(n)} .$$

Therefore, $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_{\pi(n)}$ so that the claim is established.

Now suppose that $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent. The fact that $\sum\limits_{n=1}^{\infty} |a_{\pi(n)}| = \sum\limits_{n=1}^{\infty} |a_n|$ (from the claim above) then shows that $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$ is absolutely convergent. For a given sequence $\{c_n\}_{n=1}^{\infty}$, define $c_n^+ = \max\{c_n, 0\}$ and $c_n^- = \max\{-c_n, 0\}$. Then $c_n = c_n^+ - c_n^-$ for each n $\in \mathbb{N}$. Now, since

$$0 \leqslant a_n^{\pm} \leqslant |a_n| \quad \text{and} \quad 0 \leqslant a_{\pi(n)}^{\pm} \leqslant |a_{\pi(n)}| \qquad \forall\, n \in \mathbb{N}$$

the absolute convergence of $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$ together with the comparison test show that $\sum\limits_{n=1}^{\infty} a_n^{\pm}$ and $\sum\limits_{n=1}^{\infty} a_{\pi(n)}^{\pm}$ all converge. Therefore,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} (a_n^+ - a_n^-) = \lim_{N\to\infty} \Big(\sum_{n=1}^{N} a_n^+ - \sum_{n=1}^{N} a_n^- \Big) = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- ,$$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = \lim_{N\to\infty} \sum_{n=1}^{N} (a_{\pi(n)}^+ - a_{\pi(n)}^-) = \lim_{N\to\infty} \Big(\sum_{n=1}^{N} a_{\pi(n)}^+ - \sum_{n=1}^{N} a_{\pi(n)}^- \Big) = \sum_{n=1}^{\infty} a_{\pi(n)}^+ - \sum_{n=1}^{\infty} a_{\pi(n)}^- .$$

By the claim above, we have $\sum\limits_{n=1}^{\infty} a_n^{\pm} = \sum\limits_{n=1}^{\infty} a_{\pi(n)}^{\pm}$; thus the two identities above shows that $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_{\pi(n)}$. □

Problem 3. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geqslant N$. Define

$$c_n = b_n - b_{n+1} \frac{a_{n+1}}{a_n} \qquad \forall\, n \in \mathbb{N} . \qquad (\star)$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

   Hint: Rewrite $(\star)$ as $b_n = c_n + \dfrac{a_{n+1}}{a_n} b_{n+1}$ and then obtain

$$\begin{aligned} b_N &= c_N + \frac{a_{N+1}}{a_N} b_{N+1} = c_N + \frac{a_{N+1}}{a_N}\Big(c_{N+1} + \frac{a_{N+2}}{a_{N+1}} b_{N+2}\Big) = c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N}\Big(c_{N+2} + \frac{a_{N+3}}{a_{N+2}} b_{N+3}\Big) = \cdots \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} c_{N+2} + \cdots + \frac{a_{N+n}}{a_N} c_{N+n} + \frac{a_{N+n+1}}{a_N} b_{N+n+1}\,. \end{aligned}$$

   Use the fact that $0 < r < c_n$ for all $n \geqslant N$ to conclude that

$$\sum_{k=N}^{N+n} a_k \leqslant \frac{a_N b_N}{r} \qquad \forall\, n \in \mathbb{N}\,.$$

   Note that then the sequence of partial sum of $\sum\limits_{k=1}^{\infty} a_k$ then is bounded from above $\Big($by $\sum\limits_{k=1}^{N-1} a_k + \dfrac{a_N b_N}{r}\Big)$.

2. Show that if $\sum\limits_{k=1}^{\infty} \dfrac{1}{b_k}$ diverges and $c_n \leqslant 0$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

   Hint: The fact that $c_n \leqslant 0$ for all $n \geqslant N$ implies that $b_n a_n \leqslant b_{n+1} a_{n+1}$ for all $n \geqslant N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leqslant a_n \qquad \forall\, n \geqslant N$$

   and then apply the direct comparison test to conclude that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Proof. The hints are exactly the proof. □

Problem 4. Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$. We know from class that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test)**:

   (a) If there exists a constant $\mu > 1$ such that $\dfrac{a_{n+1}}{a_n} < 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

   (b) If there exists a constant $0 < \mu < 1$ such that $\dfrac{a_{n+1}}{a_n} > 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

   Hint: Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = (n-1)a_n - n a_{n+1}$. Then $\sum\limits_{k=1}^{\infty} b_k$ is a telescoping series. For case (a), show that $\big\{n a_{n+1}\big\}_{n=N}^{\infty}$ is a positive decreasing sequence and then conclude that $\sum\limits_{k=1}^{\infty} b_k$ converges. Note that $b_n \geqslant (\mu - 1)a_n$ for all $n \geqslant N$. For case (b), show that $\big\{n a_{n+1}\big\}_{n=N}^{\infty}$ is a positive increasing sequence; thus $a_n \geqslant \dfrac{N a_{N+1}}{n-1}$ for all $n \geqslant N+1$ which implies that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Remark: 注意到 (a) 說的是如果 $\{a_n\}_{n=1}^{\infty}$ 在某項之後「遞減得夠快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 收斂。反之，如果 $\{a_n\}_{n=1}^{\infty}$「並非遞減得那麼快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 發散。

2. **(Gauss's test)**: Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$

(a) If $\mu > 1$, then $\sum\limits_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leqslant 1$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Hint: Show that if $\mu > 1$ or $\mu < 1$, one can apply Raabe's test to conclude Gauss's test. For the case $\mu = 1$, let $b_n = (n-1)\ln(n-1)$ for $n \geqslant 2$. Using the second result of Problem 3 to show the divergence of $\sum\limits_{k=1}^{\infty} a_k$ (by showing that c_n defined by $(\star)$ is non-positive for all large enough n).

Proof. 1. For each $n \in \mathbb{N}$, define $b_n = (n-1)a_n - na_{n+1}$. Then $\sum\limits_{n=1}^{\infty} b_n$ is a telescoping series. In fact,

$$\begin{aligned}\sum_{n=1}^{N} b_n &= \sum_{n=1}^{N} \big[(n-1)a_n - na_{n+1}\big] \\ &= -a_2 + (a_2 - 2a_3) + (2a_3 - 3a_4) + \cdots + \big[(N-1)a_N - Na_{N+1}\big] \\ &= -Na_{N+1}\,;\end{aligned}$$

thus $\sum\limits_{n=1}^{\infty} b_n$ converges if and only if the sequence $\{na_{n+1}\}_{n=1}^{\infty}$ converges.

(a) Suppose that there exists a constant $\mu > 1$ such that $\dfrac{a_{n+1}}{a_n} < 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$. Then $na_{n+1} < (n-\mu)a_n$ for all $n \geqslant N$ which further implies that

$$na_{n+1} - (n-1)a_n < (1-\mu)a_n < 0 \qquad \forall\, n \geqslant N\,. \tag{$\diamond$}$$

Therefore, $\{na_{n+1}\}_{n=N}^{\infty}$ is a decreasing sequence. Since $na_{n+1} > 0$, the Monotone Sequence Property of $\mathbb{R}$ implies that $\lim\limits_{n\to\infty} na_{n+1}$ exists. Therefore, $\sum\limits_{n=1}^{\infty} b_n$ exists. Note that $(\diamond)$ implies that $b_n > (\mu-1)a_n$ for all $n \geqslant N$; thus the comparison test shows that $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) Suppose that there exists a constant $\mu < 1$ such that $\dfrac{a_{n+1}}{a_n} > 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$. Then $na_{n+1} > (n-\mu)a_n$ for all $n \geqslant N$ which further implies that

$$na_{n+1} - (n-1)a_n > (1-\mu)a_n > 0 \qquad \forall\, n \geqslant N\,.$$

Therefore, $\{na_{n+1}\}_{n=N}^{\infty}$ is an increasing sequence; thus $na_{n+1} \geqslant Na_{N+1}$ for all $n \geqslant N$. This implies that $a_{n+1} \geqslant \dfrac{Na_{N+1}}{n}$ for all $n \geqslant N$. By the fact that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, the comparison test implies that $\sum\limits_{n=1}^{\infty} a_n$ diverges.

2. Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$

Suppose that $|R_n| \leqslant M$ for all $n \in \mathbb{N}$.

(a) If $\mu > 1$, then there exists $\bar{\mu}$ such that $\mu > \bar{\mu} > 1$. By the facts that $\lim\limits_{n\to\infty} \frac{M}{n^\epsilon} = 0$ and

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} = 1 - \frac{\bar{\mu}}{n} + \frac{1}{n}\Big(\mu - \bar{\mu} - \frac{R_n}{n^\epsilon}\Big)\,,$$

we find that there exists $N' \geqslant N$ such that

$$\frac{a_{n+1}}{a_n} < 1 - \frac{\bar{\mu}}{n} \qquad n \geqslant N'\,.$$

Therefore, Raabe's test shows that $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $0 < \mu < 1$, then there exists $\bar{\mu}$ such that $\mu < \bar{\mu} < 1$. By the facts that $\lim\limits_{n\to\infty} \frac{M}{n^\epsilon} = 0$ and

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} = 1 - \frac{\bar{\mu}}{n} - \frac{1}{n}\Big(\bar{\mu} - \mu + \frac{R_n}{n^\epsilon}\Big)\,,$$

we find that there exists $N' \geqslant N$ such that

$$\frac{a_{n+1}}{a_n} > 1 - \frac{\bar{\mu}}{n} \qquad n \geqslant N'\,.$$

Therefore, Raabe's test shows that $\sum\limits_{n=1}^{\infty} a_n$ diverges.

If $\mu = 1$, let $b_n = (n-1)\ln(n-1)$ for $n \geqslant 2$. Note that the function $f(x) = \frac{1}{x\ln x}$ is decreasing for $x \geqslant 3$ and

$$\int_3^\infty \frac{1}{x\ln x}\,dx = \int_{\ln 3}^\infty \frac{1}{e^u u} e^u\,du = \int_{\ln 3}^\infty \frac{1}{u}\,du = \infty\,;$$

thus the improper integral test shows that the series $\sum\limits_{n=3}^{\infty} \frac{1}{b_n}$ diverges. Moreover, if $n \geqslant N$,

$$\begin{aligned} b_n - b_{n+1}\frac{a_{n+1}}{a_n} &= (n-1)\ln(n-1) - n\ln n\Big(1 - \frac{1}{n} + \frac{R_n}{n^{1+\epsilon}}\Big) \\ &= (n-1)\ln(n-1) - (n-1)\ln n - \frac{R_n \ln n}{n^\epsilon} \\ &= \ln\Big(1 - \frac{1}{n}\Big)^{n-1} - \frac{R_n \ln n}{n^\epsilon}\,. \end{aligned}$$

Since $\lim\limits_{n\to\infty}\Big(1 - \frac{1}{n}\Big)^{n-1} = e^{-1}$ and $\lim\limits_{n\to\infty} \frac{R_n \ln n}{n^\epsilon} = 0$, we find that there exists $N' \geqslant N$ such that

$$b_n - b_{n+1}\frac{a_{n+1}}{a_n} < 0 \qquad \forall\, n \geqslant N'\,.$$

Therefore, 2 of Problem 3 shows that $\sum_{k=1}^{\infty} a_k$ diverges. □

Alternative proof of 1. (a) Suppose that there exists a constant $\mu > 1$ such that $\dfrac{a_{n+1}}{a_n} < 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$. Then

$$\frac{a_n}{a_{n+1}} > \frac{n}{n-\mu} \qquad \forall\, n \geqslant N\,.$$

Therefore,

$$n\Big(\frac{a_n}{a_{n+1}} - 1\Big) > n\Big(\frac{n}{n-\mu} - 1\Big) = \frac{n\mu}{n-\mu} \qquad \forall\, n \geqslant N\,.$$

Choose $1 < p < \mu$. Note that

$$\lim_{n\to\infty} n\Big[\frac{(n+1)^p}{n^p} - 1\Big] = \lim_{n\to\infty} \frac{(1+1/n)^p - 1^p}{1/n} = \frac{d}{dx}\Big|_{x=1} x^p = p\,;$$

thus

$$\liminf_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) \geqslant \mu > p = \limsup_{n\to\infty} n\Big[\frac{(n+1)^p}{n^p} - 1\Big]\,.$$

Therefore, (using the property of liminf and limsup) there exists $K \geqslant N$ such that

$$n\Big(\frac{a_n}{a_{n+1}} - 1\Big) > n\Big[\frac{(n+1)^p}{n^p} - 1\Big] \qquad \forall\, n \geqslant K\,;$$

thus

$$(n+1)^p a_{n+1} \leqslant n^p a_n \qquad \forall\, n \geqslant K\,.$$

The inequality above implies that the sequence $\{n^p a_n\}_{n=K}^{\infty}$ is decreasing; thus

$$n^p a_n \leqslant K^p a_K \qquad \forall\, n \geqslant K\,.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges, we conclude from the comparison test that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose that there exists a constant $0 < \mu < 1$ such that $\dfrac{a_{n+1}}{a_n} > 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$. Then

$$\frac{a_n}{a_{n+1}} < \frac{n}{n-\mu} \qquad \forall\, n \geqslant N\,.$$

Therefore,

$$n\Big(\frac{a_n}{a_{n+1}} - 1\Big) < n\Big(\frac{n}{n-\mu} - 1\Big) = \frac{n\mu}{n-\mu} \qquad \forall\, n \geqslant N\,.$$

Note that

$$\limsup_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) \leqslant \mu < 1 = \liminf_{n\to\infty} n\Big(\frac{n+1}{n} - 1\Big)\,.$$

Therefore, (using the property of liminf and limsup) there exists $K \geqslant N$ such that

$$n\Big(\frac{a_n}{a_{n+1}} - 1\Big) < n\Big(\frac{n+1}{n} - 1\Big) \qquad \forall\, n \geqslant K\,;$$

thus

$$\frac{a_{n+1}}{a_n} \geqslant \frac{n}{n+1} \qquad \forall\, n \geqslant K\,.$$

The inequality above implies that $(n+1)a_{n+1} \geqslant na_n$ for all $n \geqslant K$; thus the sequence $\{na_n\}_{n=K}^{\infty}$ is increasing. Therefore,

$$na_n \geqslant Ka_K \qquad \forall\, n \geqslant K\,.$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude from the comparison test that $\sum\limits_{n=1}^{\infty} a_n$ diverges. □

Problem 5. Complete the following.

1. Show that $\sum\limits_{k=1}^{\infty} \Big(1 - \frac{1}{\sqrt{k}}\Big)^k$ converges.
2. Show that $\sum\limits_{k=2}^{\infty} \frac{\log(k+1) - \log k}{(\log k)^2}$ converges.
3. Use Gauss's test to show that both the general harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{ak+b}$, where $a \neq 0$, and the series $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverge.
4. Show that $\sum\limits_{k=1}^{\infty} \frac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha > 1$ and diverges if $\alpha \leqslant 1$.
5. Test the following "hypergeometric" series for convergence or divergence:

   (a) $\sum\limits_{k=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)} = \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots.$

   (b) $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2\gamma \cdot (\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} + \cdots.$

Problem 6. Let $\sum\limits_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum\limits_{k=1}^{\infty} \big[1 + \operatorname{sgn}(a_k)\big] a_k$ and $\sum\limits_{k=1}^{\infty} \big[1 - \operatorname{sgn}(a_k)\big] a_k$ both diverge. Here the sign function sgn is defined by

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Proof. **Claim**: Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers. If $\{x_n\}_{n=1}^{\infty}$ converges and $\{y_n\}_{n=1}^{\infty}$ diverges, then $\{x_n \pm y_n\}_{n=1}^{\infty}$ diverges.

To see the claim, suppose the contrary that $\{x_n + y_n\}_{n=1}^{\infty}$ converges. Then Theorem 1.40 in the lecture note implies that $\{x_n + y_n - x_n\}_{n=1}^{\infty}$ converges, which contradicts the assumption that $\{y_n\}_{n=1}^{\infty}$ diverges. Similarly, $\{x_n - y_n\}_{n=1}^{\infty}$ also diverges.

Let $S_n = \sum\limits_{k=1}^{n} a_k$ and $T_n = \sum\limits_{k=1}^{n} |a_k|$. Then $\{S_n\}_{n=1}^{\infty}$ converges but $\{T_n\}_{n=1}^{\infty}$ diverges. Therefore, the claim above shows that $\{S_n \pm T_n\}_{n=1}^{\infty}$ diverges. By the fact that $|a| = \operatorname{sgn}(a)a$ for all $a \in \mathbb{R}$, we have

$$S_n \pm T_n = \sum_{k=1}^{n} (a_k \pm |a_k|) = \sum_{k=1}^{n} \big[1 \pm \operatorname{sgn}(a_k)\big] a_k$$

so we conclude the desired result. □

Problem 7. Consider the function $f(x) = \sum\limits_{k=1}^{\infty} \frac{\sin(kx)}{k}$.

1. Find the domain of f.

2. Show that for each $\varepsilon > 0$ and $0 < \delta < \pi$, there exists $N > 0$ and N depends only on ε and δ but is independent of x, such that

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \qquad \forall\, n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta]\,.$$

Proof. Let $S_n(x) = \sum_{k=1}^{n} \sin(kx)$.

1. (a) If $x = 2n\pi$ for some $n \in \mathbb{Z}$ (or $x = 0 \pmod{2\pi}$), then $S_n(x) = 0$ for all $n \in \mathbb{N}$; thus for each $x = 0 \pmod{2\pi}$, $\{S_n(x)\}_{n=1}^{\infty}$ is bounded by 1.

   (b) If $x \neq 2n\pi$ for all $n \in \mathbb{Z}$ (or $x \neq 0 \pmod{2\pi}$), then

$$\begin{aligned} 2\sin\frac{x}{2} S_n(x) &= \sum_{k=1}^{n} 2\sin\frac{x}{2}\sin(kx) = \sum_{k=1}^{n} \cos\big(k - \frac{1}{2}\big)x - \cos\big(k + \frac{1}{2}\big)x \\ &= \cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x \end{aligned}$$

   which implies that

$$\big|S_n(x)\big| \leqslant \Big|\frac{\cos\frac{x}{2} - \cos\big(n + \frac{1}{2}\big)x}{2\sin\frac{x}{2}}\Big| \leqslant \frac{1}{\big|\sin\frac{x}{2}\big|} \qquad \forall\, x \neq 0 \pmod{2\pi}\,.$$

   In either cases, for each $x \in \mathbb{R}$ there exists $M = M(x) \in \mathbb{R}$ such that $|S_n(x)| \leqslant M$. Therefore, the Dirichlet test $\big($with $a_k = \sin(kx)$ and $p_k = \dfrac{1}{k}\big)$ implies that f is defined everywhere; thus the domain of f is $\mathbb{R}$.

2. We mimic the proof of the Dirichlet test. Let $\varepsilon > 0$ and $\delta \in (0, 2\pi)$ be given. Then $\csc\dfrac{\delta}{2} > 0$; thus the Archimedean property of $\mathbb{R}$ implies that there exists $N > \dfrac{2}{\varepsilon}\csc\dfrac{\delta}{2}$. If $n \geqslant N$, $p \geqslant 0$ and $x \in [\delta, 2\pi - \delta]$ (thus $x \neq 0 \pmod{2\pi}$), then

$$\begin{aligned} \Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| &= \Big|\sum_{k=n}^{n+p} \big[S_{k+1}(x) - S_k(x)\big]\frac{1}{k}\Big| \\ &= \Big|-S_n(x)\frac{1}{n} + S_{n+1}(x)\big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + S_{n+p}(x)\big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) \\ &\quad + S_{n+p+1}(x)\frac{1}{n+p}\Big| \\ &\leqslant \frac{1}{\big|\sin\frac{x}{2}\big|}\Big[\frac{1}{n} + \big(\frac{1}{n} - \frac{1}{n+1}\big) + \cdots + \big(\frac{1}{n+p-1} - \frac{1}{n+p}\big) + \frac{1}{n+p}\Big] \\ &= \frac{2}{n\big|\sin\frac{x}{2}\big|} < \frac{\sin\frac{\delta}{2}}{\big|\sin\frac{x}{2}\big|}\varepsilon\,. \end{aligned}$$

   Since $x \in [\delta, 2\pi - \delta]$, $\sin\dfrac{x}{2}$ attains its minimum at $x = \delta$ or $2\pi - \delta$; thus

$$0 < \sin\frac{\delta}{2} \leqslant \sin\frac{x}{2} \qquad \forall\, x \in [\delta, 2\pi - \delta]\,.$$

   Therefore,

$$\Big|\sum_{k=n}^{n+p} \frac{\sin(kx)}{k}\Big| < \varepsilon \quad \text{ whenever } \quad n \geqslant N, p \geqslant 0 \text{ and } x \in [\delta, 2\pi - \delta]\,. \qquad \square$$