Exercise Problem Sets 5

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Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property, $b \in \mathbb{F}$ and $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that
   (a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.
   (b) if r, s in $\mathbb{Q}$, then $b^{r \cdot s} = (b^r)^s$.
2. For $x \in \mathbb{F}$, let $B(x) = \{b^t \in \mathbb{F} \,|\, t \in \mathbb{Q}, t \leqslant x\}$. Show that $\sup B(x)$ exists for all $x \in \mathbb{F}$, and $b^r = \sup B(r)$ if $r \in \mathbb{Q}$.
3. Define $b^x = \sup B(x)$ for $x \in \mathbb{F}$. Show that $B(x) > 0$ for all $x \in \mathbb{F}$ and the law of exponents (for exponents in $\mathbb{F}$)

   (a) if x, y in $\mathbb{F}$, then $b^{x+y} = b^x \cdot b^y$, (b) if $x, y > 0$, then $b^{x \cdot y} = (b^x)^y$,

   are also valid.
4. Show that if $x_1, x_2 \in \mathbb{F}$ and $x_1 < x_2$, then $b^{x_1} < b^{x_2}$. This implies that if x_1, x_2 are two numbers in $\mathbb{F}$ satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.
5. Let $y > 0$ be given. Show that if $u, v \in \mathbb{F}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.
6. Let $y > 0$ be given, and $A \subseteq \mathbb{F}$ be the set of all w such that $b^w < y$. Show that $\sup A$ exists and $x = \sup A$ satisfies $b^x = y$. The number x (the uniqueness is guaranteed by 4) satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

Hint: Make use of Problem 2 in Exercise 3.

Proof. We note that $\mathbb{F}$ also satisfies the Archimedean Property and the least upper bound property because of a Proposition and a Theorem that we talked about in class.

1. Note that the exponential law holds if the exponents are integers; that is,

$$b^{n+m} = b^n \cdot b^m \quad \text{and} \quad b^{nm} = (b^n)^m \qquad \forall\, n, m \in \mathbb{Z}\,.$$

   For $m, n \in \mathbb{N}$, we "define" $b^{\frac{n}{m}}$ as the n-th power of $b^{\frac{1}{m}}$; that is, $b^{\frac{n}{m}} = \big(b^{\frac{1}{m}}\big)^n$. Then for $m, n \in \mathbb{N}$,

$$\Big[\big(b^{\frac{1}{m}}\big)^n\Big]^m = \big(b^{\frac{1}{m}}\big)^{mn} = b^{\frac{mn}{m}} = b^n$$

   which implies that $\big(b^{\frac{1}{m}}\big)^n$ is the m-th root of b^n if $m, n \in \mathbb{N}$. Moreover, $\big(b^{\frac{1}{mn}}\big)^n = b^{\frac{1}{m}}$ and $\big(b^{\frac{1}{mn}}\big)^m = b^{\frac{1}{n}}$; thus we establish that

$$b^{\frac{n}{m}} = \big(b^{\frac{1}{m}}\big)^n = (b^n)^{\frac{1}{m}} \qquad \text{and} \qquad b^{\frac{1}{mn}} = \big(b^{\frac{1}{m}}\big)^{\frac{1}{n}} \qquad \forall\, m, n \in \mathbb{N}\,. \tag{$\spadesuit$}$$

Suppose that $r = \dfrac{q_1}{p_1}$ and $s = \dfrac{q_2}{p_2}$, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$. Then (♠) implies that

$$(b^r)^s = \left(b^{\frac{q_1}{p_1}}\right)^{\frac{q_2}{p_2}} = \left(b^{\frac{1}{p_1}}\right)^{\frac{q_1 q_2}{p_2}} = \left[\left(b^{\frac{1}{p_1}}\right)^{\frac{1}{p_2}}\right]^{q_1 q_2} = \left(b^{\frac{1}{p_1 p_2}}\right)^{q_1 q_2} = b^{\frac{q_1 q_2}{p_1 p_2}}$$

and

$$b^{r+s} = b^{\frac{p_2 q_1 + p_1 q_2}{p_1 p_2}} = \left(b^{\frac{1}{p_1 p_2}}\right)^{p_2 q_1 + p_1 q_2} = \left(b^{\frac{1}{p_1 p_2}}\right)^{p_2 q_1} \cdot \left(b^{\frac{1}{p_1 p_2}}\right)^{p_1 q_2} = b^{\frac{p_2 q_1}{p_1 p_2}} \cdot b^{\frac{p_1 q_2}{p_1 p_2}} = b^r \cdot b^s \,.$$

Therefore,

$$b^{r+s} = b^r \cdot b^s \quad \text{and} \quad b^{rs} = (b^r)^s \qquad \forall\, r, s \in \mathbb{Q} \ \text{ and } \ r, s > 0\,. \tag{♡}$$

For $r \in \mathbb{Q}$ and $r < 0$, we define $b^r = (b^{-r})^{-1}$. Then if $r, s \in \mathbb{Q}$ and $r, s < 0$, we have

$$b^{r+s} = (b^{-(r+s)})^{-1} = (b^{-r} \cdot b^{-s})^{-1} = (b^{-r})^{-1} \cdot (b^{-s})^{-1} = b^r \cdot b^s$$

and

$$(b^r)^s = \left[(b^{-r})^{-1}\right]^s$$

2. First we show that $x \in \mathbb{F}$, $B(x)$ is non-empty and bounded from above. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $-x < n$. Therefore, there exists a rational number $-n$ such that $-n < x$; thus $b^{-n} \in B(x)$ which implies that $B(x)$ is non-empty.

   On the other hand, the Archimedean Property implies that there exists $m \in \mathbb{N}$ such that $x < m$. By the fact that

   $$b^t \leqslant b^s \qquad \text{whenever} \quad t \leqslant s \text{ and } t, s \in \mathbb{Q}\,, \tag{$*$}$$

   we conclude that b^m is an upper bound for $B(x)$. Therefore, $B(x)$ is bounded from above. By the least upper bound property, we conclude that $\sup B(x)$ exists for all $x \in \mathbb{F}$.

   Next we show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. To see this, we note that $b^r \in B(r)$ if $r \in \mathbb{Q}$. On theother hand, $(*)$ implies that b^r is an upper bound for $B(r)$; thus $\sup B(r) = b^r$.

3. We first show that

   $$\sup(cA) = c \cdot \sup A \qquad \forall\, c > 0\,, \tag{$\star$}$$

   where $cA = \{c \cdot x \,|\, x \in A\}$. To see $(\star)$, we observe that

   $$x \in A \Rightarrow x \leqslant \sup A \Rightarrow c \cdot x \leqslant c \cdot \sup A \ \ \text{(by the compatibility of $\cdot$ and $\leqslant$)}\,;$$

   thus every element in cA is bounded from above by $c \cdot \sup A$. Therefore,

   $$\sup(cA) \leqslant c \cdot \sup A\,.$$

   On the other hand, let $\varepsilon > 0$ be given. Then there exists $x \in A$ and $x > \sup A - \dfrac{\varepsilon}{c}$. Therefore, $c \cdot x > c \cdot \sup A - \varepsilon$; thus

   $$\sup(cA) \geqslant c \cdot x > c \cdot \sup A - \varepsilon\,.$$

   Since $\varepsilon > 0$ is given arbitrarily, we find that $\sup(cA) \geqslant c \cdot \sup A$; thus $(\star)$ is concluded.

Next we show that

$$\sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant x\right\} = \inf\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\}. \tag{$\diamond$}$$

Let $S(x) = \left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\}$. If $b^t \in B(x)$, then b^t is a lower bound for $S(x)$. Therefore, $B(x)$ is a subset of the collection of all lower bounds for $S(x)$. By Problem 3 of Exercise 2,

$$\sup B(x) \leqslant \sup\left\{y \,\middle|\, y \text{ is a lower bound for } S(x)\right\} = \inf S(x).$$

Suppose that $\sup B(x) < \inf S(x)$. Since $b^{\frac{1}{n}} \searrow 1$ as $n \to \infty$ (Problem 4 of Exercise 1), there exists $n \in \mathbb{N}$ such that $\inf S(x) > b^{\frac{1}{n}} \sup B(x)$. By the fact that there exists $r \in \mathbb{Q}$ and $x \leqslant r \leqslant x + \dfrac{1}{n}$, we find that

$$\begin{aligned}\inf S(x) > b^{\frac{1}{n}} \sup B(x) &= \sup\left\{b^{r+\frac{1}{n}} \,\middle|\, r \in \mathbb{Q}, r \leqslant x\right\} = \sup\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \leqslant x + \frac{1}{n}\right\} \\ &\geqslant b^r \geqslant \inf\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \geqslant x\right\} = \inf S(x),\end{aligned}$$

a contradiction. Observe that

$$\sup A^{-1} = \big(\inf A\big)^{-1} \qquad \text{for every subset } A \text{ of } (0,\infty),$$

where $A^{-1} = \left\{t^{-1} \,\middle|\, t \in A\right\}$ and $(0,\infty)$ is the collection consisting of positive elements in $\mathbb{F}$. Therefore, $(\diamond)$ implies that for $x \in \mathbb{F}$,

$$\begin{aligned}b^{-x} &= \sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant -x\right\} = \sup\left\{b^{-t} \,\middle|\, t \in \mathbb{Q}, t \geqslant x\right\} = \Big[\inf\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \geqslant x\right\}\Big]^{-1} \\ &= (b^x)^{-1}.\end{aligned}$$

Now we show the law of exponential

$$b^x \cdot b^y = b^{x+y} \qquad \forall\, x, y \in \mathbb{F}. \tag{$\star\star$}$$

Let $x, y \in \mathbb{F}$ be given. If $t, s \in \mathbb{Q}$ and $t \leqslant x$, $s \leqslant y$, then $t + s \in \mathbb{Q}$ and $t + s \leqslant x + y$; thus

$$b^t \cdot b^s = b^{t+s} \leqslant \sup B(x+y) = b^{x+y}.$$

For any given rational $t \leqslant x$, taking the supremum of the left-hand side over all rational $s \leqslant y$ and using $(\star)$ we find that

$$b^t \cdot b^y = b^t \cdot \sup\left\{b^s \,\middle|\, s \in \mathbb{Q}, s \leqslant y\right\} \leqslant b^{x+y}.$$

Taking the supremum of the left-hand side over all rational $t \leqslant x$, using $(\star)$ again we find that

$$b^y \cdot b^x = b^y \cdot \sup\left\{b^t \,\middle|\, t \in \mathbb{Q}, t \leqslant x\right\} \leqslant b^{x+y};$$

thus we establish that

$$b^x \cdot b^y \leqslant b^{x+y} \qquad \forall\, x, y \in \mathbb{F} \tag{$\diamond\diamond$}$$

Now, note that $(\diamond\diamond)$ implies that for all $x, y \in \mathbb{F}$,

$$b^y = b^{-x+x+y} \geqslant b^{-x} \cdot b^{x+y} = (b^x)^{-1} \cdot b^{x+y} \geqslant (b^x)^{-1} \cdot b^x \cdot b^y = b^y\,.$$

The inequality above is indeed an equality and we obtain that

$$b^y = b^{-x} b^{x+y} \qquad \forall\, x, y \in \mathbb{F}\,.$$

This is indeed $(\star\star)$ because of that $b^{-x} = (b^x)^{-1}$.

Next we show that $(b^x)^y = \sup B(x \cdot y)$ for all $x > 0$ and $y \in \mathbb{F}$. For $z > 0$, define $A(z) = \big\{ s \in \mathbb{F} \,\big|\, s \in \mathbb{Q}, 0 < s \leqslant z \big\}$. Note that if $z > 0$, then $b^z = \sup A(z)$. Since for $x > 0$, we have $b^x > 1$; thus for $x, y > 0$,

$$(b^x)^y = \sup \big\{ (b^x)^t \,\big|\, t \in \mathbb{Q}, 0 < t \leqslant y \big\} = \sup_{t \in A(y)} (b^x)^t = \sup_{t \in A(y)} \big(\sup_{s \in A(x)} b^s \big)^t\,.$$

By Problem 4 of Exercise 2,

$$\sup_{t \in A(y)} \big(\sup_{s \in A(x)} b^s \big)^t = \sup_{(t,s) \in A(y) \times A(x)} (b^s)^t = \sup_{(t,s) \in A(y) \times A(x)} b^{st} = b^{\sup_{(t,s) \in A(y) \times A(x)} ts} = b^{xy}\,.$$

4. Let $x_1 < x_2$ be given. Then **AP** implies that there exists $r, s \in \mathbb{Q}$ such that $x_1 < r < s < x_2$. Therefore, $B(x_1) \subseteq B(r) \subseteq B(s) \subseteq B(x_2)$; thus

$$b^{x_1} = \sup B(x_1) \leqslant \sup B(r) \leqslant \sup B(s) \leqslant \sup B(x_2) = b^{x_2}\,.$$

Since $B(r) = b^r$ and $B(s) = b^s$, we must have $B(r) < B(s)$; thus 4 is concluded.

5. Since $\dfrac{y}{b^u} > 1$ and $\dfrac{b^v}{y} > 1$, by the fact that $b^{\frac{1}{n}} \to 1$ as $n \to \infty$, there exist $N_1, N_2 > 0$ such that

$$\big| b^{\frac{1}{n}} - 1 \big| < \frac{y}{b^u} - 1 \;\text{ whenever }\; n \geqslant N_1 \quad\text{and}\quad \big| b^{\frac{1}{n}} - 1 \big| < \frac{b^v}{y} - 1 \;\text{ whenever }\; n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. For $n \geqslant N$, we have $b^{\frac{1}{n}} < \dfrac{y}{b^u}$ and $b^{\frac{1}{n}} < \dfrac{b^v}{y}$ or equivalently,

$$b^{u+\frac{1}{n}} < y \quad\text{and}\quad b^{v-\frac{1}{n}} > y \qquad \forall\, n \geqslant N\,.$$

6. Let $A = \big\{ w \in \mathbb{F} \,\big|\, b^w < y \big\}$. Since $b > 1$, 2 of Problem 4 in Exercise 1 implies that

$$b^n > 1 + n(b-1) \quad\text{ whenever }\quad n \geqslant 2\,. \tag{$\star\star\star$}$$

By **AP**, there exists $N \geqslant 2$ such that $1 + N(b-1) > y$; thus A is bounded from above by N. Moreover, there exists $M \geqslant 2$ such that

$$1 + M(b-1) > \frac{1}{y}\,;$$

thus $(\star\star\star)$ implies that $b^{-M} < y$ or $-N \in A$. Therefore, A is non-empty. By **LUBP**, we conclude that $\sup A$ exists.

Let $x = \sup A$. Then $x + \dfrac{1}{n} \notin A$; thus $b^{x+\frac{1}{n}} \geqslant y$ for all $n \in \mathbb{N}$. Since $b^{\frac{1}{n}} \to 1$ sa $n \to \infty$, we find that

$$b^x = b^x \lim_{n\to\infty} b^{\frac{1}{n}} = \lim_{n\to\infty} b^{x+\frac{1}{n}} \geqslant y\,.$$

On the other hand, 4 implies that $x - \dfrac{1}{n} \in A$; thus $b^{x-\frac{1}{n}} > y$ for all $n \in \infty$ and we have

$$b^x = b^x \lim_{n\to\infty} b^{-\frac{1}{n}} = \lim_{n\to\infty} b^{x-\frac{1}{n}} \leqslant y\,.$$

Therefore, $b^x = y$. □

Problem 2. Let $(\mathbb{F}, +\cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property. In this problem we prove the Intermediate Value Theorem:

> Let $f : [a, b] \to \mathbb{F}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> If $f(a)f(b) < 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Complete the following.

1. W.L.O.G, we can assume that $f(a) < 0$. Define the set $S = \{x \in [a, b] \,|\, f(x) > 0\}$. Show that $\inf S$ exists.
2. Let $c = \inf S$. Show that $f(c) \geqslant 0$.
3. Conclude that $f(c) \leqslant 0$ as well.

Hint:

1. Show that S is non-empty and bounded from below and note that **MSP** $\Leftrightarrow$ **LUBP**.
2. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in S such that $c_n \to c$ as $n \to \infty$.
3. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in $[a, c)$ such that $c_n \to c$ as $n \to \infty$.

Proof. 1. Since $f(b) > 0$, $b \in S$. Moreover, a is a lower bound for S; thus S is non-empty and bounded from below. Since **MSP** $\Leftrightarrow$ **LUBP**, $\inf S \in \mathbb{F}$ exists.

2. Let $c = \inf S$. For each $n \in \mathbb{N}$, there exists $c_n < c + \dfrac{1}{n}$ and $c_n \in S$. Then $f(c_n) > 0$ for all $n \in \mathbb{N}$ and

$$c \leqslant c_n < c + \frac{1}{n} \qquad \forall\, n \in \mathbb{N}\,.$$

Then the Sandwich Lemma implies that $c_n \to c$ as $n \to \infty$. By the continuity of f,

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \geqslant 0\,.$$

3. By 2, $a \neq c$. Consider the sequence $\{c_n\}_{n=1}^{\infty}$ defined by $c_n = c - \dfrac{c-a}{n}$. Then $\{c_n\}_{n=1}^{\infty} \subseteq [a, c)$. Moreover, by the fact that $c = \inf S$ and $c_n < c$, $c_n \notin S$ for all $n \in \mathbb{N}$. Therefore, $f(c_n) \leqslant 0$ for all $n \in \mathbb{N}$. Since $c_n \to c$ as $n \to \infty$, by the continuity of f we find that

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \leqslant 0 . \qquad \square$$

Problem 3. Let $(\mathbb{F}, +\cdot, \leqslant)$ be an ordered field satisfying the monotone sequence property. In this problem we prove the Extreme Value Theorem:

> Let $a, b \in \mathbb{F}$, $a < b$ and $f : [a, b] \to \mathbb{F}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> Then there exist $c, d \in [a, b]$ such that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Complete the following.

1. Show that there exist sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ in $[a, b]$ such that

$$\lim_{n\to\infty} f(c_n) = \sup_{x\in[a,b]} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f(d_n) = \inf_{x\in[a,b]} f(x) .$$

2. Extract convergent subsequences $\{c_{n_k}\}_{k=1}^{\infty}$ and $\{d_{n_k}\}_{k=1}^{\infty}$ with limit c and d, respectively. Show that $c, d \in [a, b]$.

3. Show that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Hint: For 2, note that **MSP** $\Rightarrow$ **BWP**.

Proof. It suffices to show the case of $\sup\limits_{x\in[a,b]} f(x)$ since $\inf\limits_{x\in[a,b]} f(x) = -\sup\limits_{x\in[a,b]} (-f)(x)$ by Problem 1 of Exercise 2.

1. We first show that $f([a, b])$ is bounded. Suppose the contrary that $f([a, b])$ is not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $\big|f(x_n)\big| > n$. Since $\{x_n\}_{n=1}^{\infty} \subseteq [a, b]$, $\{x_n\}_{n=1}^{\infty}$ is bounded. By the fact that **MSP** $\Rightarrow$ **BWP**, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. By the continuity of f, $\big\{f(x_{n_k})\big\}_{k=1}^{\infty}$ is also convergent; thus Proposition 1.39 in the lecture note implies that $\big\{f(x_{n_k})\big\}_{k=1}^{\infty}$ is bounded, a contradiction to that $\big|f(x_{n_k})\big| \geqslant n_k \geqslant k$ for all $k \in \mathbb{N}$.

   Since $f([a, b])$ is bounded, $M = \sup f([a, b]) = \sup\limits_{x\in[a,b]} f(x)$ exists. For each $n \in \mathbb{F}$, there exists $c_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(c_n) \leqslant M .$$

   By the Sandwich Lemma, $\lim\limits_{n\to\infty} f(c_n) = M = \sup\limits_{x\in[a,b]} f(x)$.

2. Since $\{c_n\}_{n=1}^{\infty} \subseteq [a,b]$, $\{c_n\}_{n=1}^{\infty}$ is bounded. By the fact that **MSP** $\Rightarrow$ **BWP**, there exists a convergent subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ with limit c. Since $a \leqslant c_{n_k} \leqslant b$ for all $k \in \mathbb{N}$, by a Proposition that we talked about in class we conclude that $a \leqslant c \leqslant b$.

3. Since $c_{n_k} \to c$ as $k \to \infty$, the continuity of f implies that

$$f(c) = f(\lim_{k\to\infty} c_{n_k}) = \lim_{k\to\infty} f(c_{n_k}) = \sup_{x\in[a,b]} f(x)\,. \qquad \square$$

Problem 4. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leqslant \liminf_{n\to\infty}(x_n+y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\leqslant \limsup_{n\to\infty}(x_n+y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\
\big(\liminf_{n\to\infty}|x_n|\big)\big(\liminf_{n\to\infty}|y_n|\big) &\leqslant \liminf_{n\to\infty}|x_ny_n| \leqslant \big(\liminf_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big) \\
&\leqslant \limsup_{n\to\infty}|x_ny_n| \leqslant \big(\limsup_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big)\,.
\end{aligned}$$

Give examples showing that the equalities are generally not true.

Proof. 1. Let $k \in \mathbb{N}$ be fixed. Note that for $n \geqslant k$, we have

$$\inf_{n\geqslant k}(x_n+y_n) \leqslant x_n + y_n \leqslant \sup_{n\geqslant k}(x_n+y_n)\,.$$

Note that the LHS and the RHS are functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k}\Big[\inf_{n\geqslant k}(x_n+y_n) - y_n\Big] \leqslant \inf_{n\geqslant k} x_n \leqslant \inf_{n\geqslant k}\Big[\sup_{n\geqslant k}(x_n+y_n) - y_n\Big]$$

which further shows that

$$\inf_{n\geqslant k}(x_n+y_n) - \sup_{n\geqslant k} y_n \leqslant \inf_{n\geqslant k} x_n \leqslant \sup_{n\geqslant k}(x_n+y_n) - \sup_{n\geqslant k} y_n\,.$$

Therefore,

$$\inf_{n\geqslant k}(x_n+y_n) \leqslant \inf_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n \leqslant \sup_{n\geqslant k}(x_n+y_n) \qquad \forall\, k \in \mathbb{N}\,,$$

and the first inequality follows from the fact that

$$\inf_{n\geqslant k} x_n + \inf_{n\geqslant k} y_n \leqslant \inf_{n\geqslant k}(x_n+y_n) \leqslant \inf_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n \leqslant \sup_{n\geqslant k}(x_n+y_n) \leqslant \sup_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n$$

for each $k \in \mathbb{N}$.

2. Let $k \in \mathbb{N}$ be fixed. Note that for $n \geqslant k$, we have

$$\inf_{n\geqslant k}\Big[|x_n|\big(|y_n| + \frac{1}{k}\big)\Big] \leqslant |x_n|\big(|y_n| + \frac{1}{k}\big) \leqslant \sup_{n\geqslant k}\Big[|x_n|\big(|y_n| + \frac{1}{k}\big)\Big]\,.$$

Note that the LHS and the RHS for functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k} \frac{\inf_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}} \leqslant \inf_{n\geqslant k}|x_n| \leqslant \inf_{n\geqslant k} \frac{\sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}}\,.$$

By the fact that

$$\inf_{n\geqslant k}\frac{1}{|y_n|+\frac{1}{k}}=\frac{1}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\,,$$

we find that

$$\frac{\inf\limits_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\leqslant\inf_{n\geqslant k}|x_n|\leqslant\inf_{n\geqslant k}\frac{\sup\limits_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]}{\sup\limits_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)}\,;$$

thus

$$\inf_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]\leqslant\inf_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\leqslant\sup_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]\,.$$

The second inequality follows from the fact that

$$\begin{aligned}\inf_{n\geqslant k}|x_n|\inf_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\leqslant\inf_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]\leqslant\inf_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\\ \leqslant\sup_{n\geqslant k}\big[|x_n|\big(|y_n|+\frac{1}{k}\big)\big]\leqslant\sup_{n\geqslant k}|x_n|\sup_{n\geqslant k}\big(|y_n|+\frac{1}{k}\big)\end{aligned}$$

for each $k\in\mathbb{N}$, and passing to the limit as $k\to\infty$.

3. Let $x_n=2+\sin n$ and $y_n=2+\cos n$. Then $x_n,y_n>0$, and

$$\liminf_{n\to\infty}x_n=\liminf_{n\to\infty}y_n=1\,,\quad\limsup_{n\to\infty}x_n=\limsup_{n\to\infty}y_n=3\,.$$

By Problem 3, the set $\big\{x\in[0,2\pi]\,\big|\,x=k\pmod{2\pi}\text{ for some }k\in\mathbb{N}\big\}$ is dense in $[0,2\pi]$; thus for each $\theta\in[0,2\pi]$ there exists an increasing sequence $\{k_j\}_{j=1}^\infty\subseteq\mathbb{N}$ such that $x_{k_j}=k_j\pmod{2\pi}$ and $\big\{x_{k_j}\big\}_{j=1}^\infty$ converges to θ. This implies that for each $\theta\in[-1,1]$, there exists a subsequence $\{\cos k_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty}\cos n_j=\cos\theta\quad\text{and}\quad\lim_{j\to\infty}\sin n_j=\sin\theta\,.$$

Therefore, we have

$$\liminf_{n\to\infty}(x_n+y_n)=4-\sqrt{2}\,,\quad\limsup_{n\to\infty}(x_n+y_n)=4+\sqrt{2}\,,$$

and

$$\liminf_{n\to\infty}x_ny_n=\frac{9}{2}-2\sqrt{2}\,,\quad\limsup_{n\to\infty}x_ny_n=\frac{9}{2}+2\sqrt{2}\,.$$

Therefore,

$$\begin{aligned}\liminf_{n\to\infty}x_n+\liminf_{n\to\infty}y_n<\liminf_{n\to\infty}(x_n+y_n)<\liminf_{n\to\infty}x_n+\limsup_{n\to\infty}y_n\\ <\limsup_{n\to\infty}(x_n+y_n)<\limsup_{n\to\infty}x_n+\limsup_{n\to\infty}y_n\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty}x_n\cdot\liminf_{n\to\infty}y_n<\liminf_{n\to\infty}(x_ny_n)<\liminf_{n\to\infty}x_n\cdot\limsup_{n\to\infty}y_n\\ <\limsup_{n\to\infty}(x_ny_n)<\limsup_{n\to\infty}x_n\cdot\limsup_{n\to\infty}y_n\,.\end{aligned}$$

Therefore, the equalities are generally not true. □

Problem 5. Prove that

$$\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\leqslant\liminf_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\,.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty}\sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Proof. W.L.O.G. we can assume that $\liminf\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}>0$ and $\limsup\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}<\infty$. Let $a=\liminf\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}$ and $b=\limsup\limits_{n\to\infty}\dfrac{|x_{n+1}|}{|x_n|}$, and $\varepsilon>0$ be given such that $a-\varepsilon>0$. Then there exists $N>0$ such that

$$a-\varepsilon<\frac{|x_{n+1}|}{|x_n|}<b+\varepsilon\qquad\forall\,n\geqslant N\,.$$

Therefore,

$$(a-\varepsilon)|x_n|<|x_{n+1}|<(b+\varepsilon)|x_n|\qquad\forall\,n\geqslant N$$

which implies that if $n>N$,

$$|x_n|>(a-\varepsilon)|x_{n-1}|>(a-\varepsilon)^2|x_{n-2}|>\cdots>(a-\varepsilon)^{n-N}|x_N|$$

and

$$|x_n|<(b+\varepsilon)|x_{n-1}|<(b+\varepsilon)^2|x_{n-2}|<\cdots<(b+\varepsilon)^{n-N}|x_N|\,.$$

The inequality above implies that

$$(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}<\sqrt[n]{|x_n|}<(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\,;$$

thus

$$\liminf_{n\to\infty}\Big[(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big]\leqslant\liminf_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\Big[(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big]\,.$$

By Problem 4 of Exercise 1, $\lim\limits_{n\to\infty}b^{\frac{1}{n}}=1$ for all $b>0$. Therefore,

$$\liminf_{n\to\infty}\Big[(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big]=\lim_{n\to\infty}(a-\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}=a-\varepsilon=\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}-\varepsilon$$

and

$$\limsup_{n\to\infty}\Big[(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}\Big]=\lim_{n\to\infty}(b+\varepsilon)^{1-\frac{N}{n}}\sqrt[n]{|x_N|}=b+\varepsilon=\limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}+\varepsilon\,.$$

Since the inequality above holds for all $\varepsilon>0$, we conclude that

$$\liminf_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\leqslant\liminf_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\sqrt[n]{|x_n|}\leqslant\limsup_{n\to\infty}\frac{|x_{n+1}|}{|x_n|}\,.$$

Let $\{x_n\}_{n=1}^\infty$ be a real sequence defined by

$$x_n=\begin{cases}2^{-n} & \text{if } n \text{ is odd}\,,\\ 4^{-n} & \text{if } n \text{ is even}\,,\end{cases}$$

or $x_n = (3 + (-1)^n)^{-n}$. Then $\sqrt[n]{|x_n|} = 3 + (-1)^n$ which shows that

$$\liminf_{n\to\infty} \sqrt[n]{|x_n|} = \frac{1}{4} \qquad \text{and} \qquad \limsup_{n\to\infty} \sqrt[n]{|x_n|} = \frac{1}{2}\,.$$

To compute the limit superior and limit inferior of $\dfrac{|x_{n+1}|}{|x_n|}$, we define

$$y_n = \frac{|x_{n+1}|}{|x_n|} = \frac{(3 + (-1)^{n+1})^{-n-1}}{(3 + (-1)^n)^{-n}} = \frac{1}{3 - (-1)^n}\Big(\frac{3 - (-1)^n}{3 + (-1)^n}\Big)^{-n}$$

and observe that $\lim\limits_{n\to\infty} y_{2n} = 0$ and $\lim\limits_{n\to\infty} y_{2n+1} = \infty$. Since $y_n \in [0, \infty)$, we conclude that 0 is the smallest cluster point of $\{y_n\}_{n=1}^{\infty}$ and ∞ is the largest "cluster point" of $\{y_n\}_{n=1}^{\infty}$. This shows that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \infty\,. \qquad \square$$

Problem 6. Find $\limsup\limits_{n\to\infty} \cos n$ and $\liminf\limits_{n\to\infty} \cos n$.

Hint: First show that for all irrational α, the set

$$S = \big\{x \in [0,1] \,\big|\, x = k\alpha \pmod 1 \text{ for some } k \in \mathbb{N}\big\}$$

is dense in $[0, 1]$; that is, for all $y \in [0, 1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y - \varepsilon, y + \varepsilon)$. Then choose $\alpha = \dfrac{1}{2\pi}$ to conclude that

$$T = \big\{x \in [0, 2\pi] \,\big|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\big\}$$

is dense in $[0, 2\pi]$. To prove that S is dense in $[0, 1]$, you might want to consider the following set

$$S_k = \big\{x \in [0,1] \,\big|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\big\}$$

Note that there must be two points in S_k whose distance is less than $\dfrac{1}{k}$. What happened to (the multiples of) the difference of these two points?

Proof. Define $S_k = \big\{x \in [0,1] \,\big|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\big\}$. Let $1 \leqslant \ell_1, \ell_2 \leqslant k+1$, and $x, y \in [0, 1]$ satisfying that $x = \ell_1\alpha \pmod 1$ and $y = \ell_2\alpha \pmod 1$. Then by the fact that $\alpha \notin \mathbb{Q}$,

$$x = y \quad \Leftrightarrow \quad \ell_1\alpha = \ell_2\alpha \pmod 1 \quad \Leftrightarrow \quad (\ell_1 - \ell_2)\alpha \in \mathbb{Z} \quad \Leftrightarrow \quad \ell_1 - \ell_2 = 0\,.$$

Therefore, there are $(k + 1)$ distinct points in S_k (this also shows that each $k \in \mathbb{N}$ corresponds to different point $x = k\alpha \pmod 1$ in S). Moreover, $x \notin \mathbb{Q}$ if $x \in S_k$. By the pigeonhole principle, there exist x, y in S_k satisfying that $0 < |x - y| < \dfrac{1}{k}$.

Let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \varepsilon$. By the discussion above, there exist $x, y \in S_n$ such that $0 < |x - y| < \varepsilon$. Suppose that $x = n_1\alpha \pmod 1$ and $y = n_2\alpha \pmod 1$, and define $m = |n_1 - n_2|$. The point $z \in [0, 1]$ satisfying $z = m\alpha \pmod 1$ has the property that $z \in (0, \varepsilon) \cup (1 - \varepsilon, 1)$. Therefore,

$$(\forall\, \varepsilon > 0)(\exists\, x \in S)\big(x \in (0, \varepsilon) \cup (1 - \varepsilon, 1)\big)\,.$$

Let $y \in [0, 1]$ and $\varepsilon > 0$ be given. The discussion above provides an $x \in (0, 1)$ such that $x = k\alpha$ (mod 1) for some $k \in \mathbb{N}$ and $x \in (0, \varepsilon) \cup (1 - \varepsilon, 1)$. Then some constant multiple of x must belong to $(y - \varepsilon, y + \varepsilon)$. If $\ell x \in (y - \varepsilon, y + \varepsilon)$, then $z = k\ell\alpha$ (mod 1) in $(y - \varepsilon, y + \varepsilon)$. This shows that S is dense in $[0, 1]$.

Having established that S is dense in $[0, 1]$, we find that T is dense in $[0, 2\pi]$. Therefore, for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{m_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ such that $x_{m_j} = m_j \pmod{2\pi}$ and $\{x_{m_j}\}_{j=1}^{\infty} \subseteq [0, 2\pi]$ converges to θ. In particular, for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{m_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ such that

$$\lim_{j \to \infty} \cos m_j = \cos\theta \qquad \text{and} \qquad \lim_{j \to \infty} \sin m_j = \sin\theta\,;$$

thus we conclude that $\limsup_{m \to \infty} \cos m = 1$ and $\liminf_{m \to \infty} \cos m = -1$. □