Exercise Problem Sets 4

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Problem 1. Let A be a set, and $f, g : A \to \mathbb{R}$ be two functions. Let $h = \max\{f, g\}$; that is,

$$h(x) = \max\big\{f(x), g(x)\big\} \qquad \forall\, x \in A\,.$$

Show that

$$\sup_{x\in A} h(x) = \max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\}\,.$$

Generalize the result above to the following: if $f_1, \cdots, f_n : A \to \mathbb{R}$ are real-valued functions, then

$$\sup_{x\in A} \max\big\{f_1(x), \cdots, f_n(x)\big\} = \max\Big\{\sup_{x\in A} f_1(x), \sup_{x\in A} f_2(x), \cdots, \sup_{x\in A} f_n(x)\Big\}\,.$$

Can one conclude that if $f_n : A \to \mathbb{R}$ is a sequence of functions, then

$$\sup_{x\in A} \sup\big\{f_1(x), \cdots, f_n(x), \cdots\big\} = \sup\Big\{\sup_{x\in A} f_1(x), \sup_{x\in A} f_2(x), \cdots, \sup_{x\in A} f_n(x), \cdots\Big\}\,.$$

Proof. First, by the definition of h,

$$f(x) \leqslant h(x) \qquad \forall\, x \in A \qquad \text{and} \qquad g(x) \leqslant h(x) \qquad \forall\, x \in A\,.$$

Therefore, by the fact that $h(x) \leqslant \sup\limits_{x\in A} h(x)$, we find that

$$f(x) \leqslant \sup_{x\in A} h(x) \quad \text{and} \quad g(x) \leqslant \sup_{x\in A} h(x) \qquad \forall\, x \in A\,.$$

The inequalities above shows that $\sup\limits_{x\in A} h(x)$ is an upper bound for the range of f and g; thus

$$\sup_{x\in A} f(x) \leqslant \sup_{x\in A} h(x) \qquad \text{and} \qquad \sup_{x\in A} g(x) \leqslant \sup_{x\in A} h(x)\,.$$

Therefore,

$$\max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\} \leqslant \sup_{x\in A} h(x)\,. \tag{$\star$}$$

Next, we show the revered inequality.

1. Suppose that $\sup\limits_{x\in A} h(x) = \infty$. Then h is not bounded from above; thus f or g is not bounded from above. In fact, if $f(x) \leqslant M$ and $g(x) \leqslant N$ for all $x \in A$, then $h(x) = \max\{f(x), g(x)\} \leqslant \max\{M, N\}$ for all $x \in A$ which shows that h is bounded from above, a contradiction. Therefore, $\sup\limits_{x\in A} f(x) = \infty$ or $\sup\limits_{x\in A} g(x) = \infty$ so that

$$\max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\} = \infty$$

which shows that

$$\max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\} \geqslant \sup_{x\in A} h(x)\,.$$

2. Suppose that $\sup\limits_{x\in A} h(x) = M \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Then there exists $x_0 \in A$ such that

$$M - \varepsilon < h(x_0) = \max\big\{f(x_0), g(x_0)\big\}.$$

Therefore, the fact $f(x_0) \leqslant \sup\limits_{x\in A} f(x)$ and $g(x_0) \leqslant \sup\limits_{x\in A} g(x)$ shows that

$$M - \varepsilon < \max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\}.$$

The inequality above holds for all $\varepsilon > 0$; thus

$$\sup_{x\in A} h(x) = M \leqslant \max\Big\{\sup_{x\in A} f(x), \sup_{x\in A} g(x)\Big\}.$$

In either case we have shown that $\sup\limits_{x\in A} h(x) = M \leqslant \max\Big\{\sup\limits_{x\in A} f(x), \sup\limits_{x\in A} g(x)\Big\}$; thus combining with $(\star)$ we conclude the desired identity.

Next we show that

$$\sup_{x\in A} \max\big\{f_1(x), \cdots, f_n(x)\big\} = \max\Big\{\sup_{x\in A} f_1(x), \sup_{x\in A} f_2(x), \cdots, \sup_{x\in A} f_n(x)\Big\}. \qquad (\star\star)$$

We note that for each $n \geqslant 3$,

$$\max\big\{f_1(x), \cdots, f_n(x)\big\} = \max\Big\{\max\big\{f_1(x), \cdots, f_{n-1}(x)\big\}, f_n(x)\Big\} \qquad \forall\, x \in A. \qquad (\star\star\star)$$

In fact, for a fixed $x \in A$ suppose that $f_j(x) = \max\big\{f_1(x), \cdots, f_n(x)\big\}$.

1. $j \neq n$: In this case $f_j(x) = \max\big\{f_1(x), \cdots, f_{n-1}(x)\big\}$ and $f_j(x) \geqslant f_n(x)$. Therefore,

$$\begin{aligned}\max\big\{f_1(x), \cdots, f_n(x)\big\} &= f_j(x) = \max\big\{f_j(x), f_n(x)\big\}\\ &= \max\Big\{\max\big\{f_1(x), \cdots, f_{n-1}(x)\big\}, f_n(x)\Big\}.\end{aligned}$$

2. $j = n$: If this case $f_n(x) \geqslant \max\big\{f_1(x), \cdots, f_{n-1}(x)\big\}$; thus

$$\max\big\{f_1(x), \cdots, f_n(x)\big\} = f_n(x) = \max\Big\{\max\big\{f_1(x), \cdots, f_{n-1}(x)\big\}, f_n(x)\Big\}.$$

This establishes $(\star\star\star)$.

Now we prove $(\star\star)$. From the argument above we find that $(\star\star)$ holds for the case $n = 2$. Suppose that $(\star\star)$ holds for the case $n = m$. If $n = m + 1$, by $(\star\star\star)$ we find that

$$\max\big\{f_1(x), \cdots, f_{m+1}(x)\big\} = \max\Big\{\max\big\{f_1(x), \cdots, f_m(x)\big\}, f_{m+1}(x)\Big\} \qquad \forall\, x \in A;$$

thus

$$\begin{aligned}\sup_{x\in A} \max\big\{f_1(x), \cdots, f_{m+1}(x)\big\} &= \sup_{x\in A} \max\Big\{\max\big\{f_1(x), \cdots, f_m(x)\big\}, f_{m+1}(x)\Big\}\\ &= \max\Big\{\sup_{x\in A} \max\big\{f_1(x), \cdots, f_m(x)\big\}, \sup_{x\in A} f_{m+1}(x)\Big\}\end{aligned}$$

and the assumption that $(\star\star)$ holds for the case $n = m$ further implies that

$$\begin{aligned}\sup_{x\in A}\max\big\{f_1(x),\cdots,f_{m+1}(x)\big\} &= \max\Big\{\max\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_m(x)\Big\},\sup_{x\in A}f_{m+1}(x)\Big\}\\ &= \max\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_{m+1}(x)\Big\}.\end{aligned}$$

Therefore, $(\star\star)$ holds for the case $n = m + 1$. By induction, $(\star\star)$ holds for all $n \geqslant 2$.

Finally, we note that

$$f_j(x) \leqslant \sup_{y\in A} f_j(y) \leqslant \sup\Big\{\sup_{y\in A}f_1(y),\cdots,\sup_{y\in A}f_n(y),\cdots\Big\} \quad \forall\, x\in A \text{ and } j\in\mathbb{N}.$$

This implies that

$$\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\} \leqslant \sup\Big\{\sup_{y\in A}f_1(y),\cdots,\sup_{y\in A}f_n(y),\cdots\Big\} \qquad \forall\, x\in A;$$

thus

$$\begin{aligned}\sup_{x\in A}\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\} &\leqslant \sup\Big\{\sup_{y\in A}f_1(y),\cdots,\sup_{y\in A}f_n(y),\cdots\Big\}\\ &= \sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\}.\end{aligned}$$

Now we prove the reverse inequality. Let $S = \sup_{x\in A}\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\}$.

1. $S\in\mathbb{R}$: Let $\varepsilon > 0$ be given. By the definition of supremum, there exists $x\in A$ such that

   $$S \geqslant \sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\} > S - \frac{\varepsilon}{2}.$$

   Then $\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\}\in\mathbb{R}$; thus there exists $j\in\mathbb{N}$ such that

   $$f_j(x) > \sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\} - \frac{\varepsilon}{2} > S - \varepsilon.$$

   Therefore, $\sup_{x\in A} f_j(x) \geqslant S-\varepsilon$ which implies that

   $$\sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\} \geqslant S - \varepsilon.$$

   Since $\varepsilon > 0$ is given arbitrarily, we find that

   $$\sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\} \geqslant S = \sup_{x\in A}\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\}.$$

2. $S=\infty$: Let $M > 0$ be given. Then there exists $x\in A$ such that

   $$\sup\big\{f_1(x),\cdots,f_n(x),\cdots\big\} > M$$

   which further implies that there exists $j\in\mathbb{N}$ such that $f_j(x) > M$. Therefore, $\sup_{x\in A} f_j(x) \geqslant M$; thus

   $$\sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\} \geqslant M.$$

   Since M is given arbitrarily, we conclude that

   $$\sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\} = \infty = S.$$

In either case we establish that $\sup_{x\in A}\sup\{f_1(x),\cdots,f_n(x),\cdots\}\geqslant S$; thus

$$\sup_{x\in A}\sup\{f_1(x),\cdots,f_n(x),\cdots\}=\sup\Big\{\sup_{x\in A}f_1(x),\cdots,\sup_{x\in A}f_n(x),\cdots\Big\}. \qquad \square$$

Problem 2. Let $(\mathbb{F},+,\cdot,\leqslant)$ be an Archimedean ordered field. A number $x\in\mathbb{F}$ is called an ***accumulation point*** of a set $A\subseteq\mathbb{F}$ if for all $\delta>0$, $(x-\delta,x+\delta)$ contains at least one point of A distinct from x. In logic notation,

$$x \text{ is an accumulation point of } A \quad\Leftrightarrow\quad (\forall\,\delta>0)\big(A\cap(x-\delta,x+\delta)\backslash\{x\}\neq\varnothing\big).$$

1. Show that if $\{x_n\}_{n=1}^\infty$ is a sequence in $\mathbb{F}$ so that $x_i\neq x_j$ for all $i,j\in\mathbb{N}$ and $A=\{x_k\,|\,k\in\mathbb{N}\}$, then x is an accumulation of A if and only if x is a cluster point of $\{x_n\}_{n=1}^\infty$.

2. How about if the condition $x_i\neq x_j$ for all $i,j\in\mathbb{N}$ is removed? Is the statement in 1 still valid?

Proof. 1. We show that

$$x \text{ is an accumulation point of } A \text{ if and only if } (\forall\,\delta>0)\big(\#(A\cap(x-\delta,x+\delta))=\infty\big).$$

The direction "$\Leftarrow$" is trivial since if $\#(A\cap(x-\delta,x+\delta))=\infty$, $A\cap(x-\delta,x+\delta)$ contains some point distinct from x.

($\Rightarrow$) Let $\delta_1=1$, by the definition of the accumulation points, there exists $x_1\in A\cap(x-\delta_1,x+\delta_1)$ and $x_1\neq x$. Define $\delta_2=\min\{|x_1-x|,\frac{1}{2}\}$. Then $\delta_2>0$; thus there exists $x_2\in A\cap(x-\delta_2,x+\delta_2)$ and $x_2\neq x$. We continue this process and obtain a sequence $\{x_n\}_{n=1}^\infty\subseteq A\backslash\{x\}$ satisfying that

$$x_1\in A\cap(x-1,x+1),\quad x_n\in A\cap(x-\delta_n,x+\delta_n) \text{ with } \delta_n=\min\Big\{|x-x_{n-1}|,\frac{1}{n}\Big\}.$$

By Archimedean property, $\{x_n\}_{n=1}^\infty$ converges to x since $|x-x_n|<\delta_n\leqslant\frac{1}{n}$. Let $\delta>0$ be given. There exists $N>0$ such that $\frac{1}{N}<\delta$; thus

$$A\cap(x-\delta,x+\delta)\supseteq A\cap\big(x-\frac{1}{N},x+\frac{1}{N}\big)\supseteq\{x_N,x_{N+1},x_{N+2},\cdots\}.$$

Since $x_i\neq x_j$ for all $i,j\in\mathbb{N}$, we must have $\#\big(A\cap(x-\delta,x+\delta)\big)=\infty$. $\square$

Problem 3. Let $(\mathbb{F},+,\cdot,\leqslant)$ be an ordered field, and $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{F}$. Show that $\{x_n\}_{n=1}^\infty$ converges if and only if every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges.

Proof. By Proposition 1.60 in the lecture note, it suffices to prove the direction "$\Leftarrow$". We show that if every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges, then every proper subsequence of $\{x_n\}_{n=1}^\infty$ converges to identical limit. Suppose the contrary that there exist two subsequence $\{x_{n_k}\}_{k=1}^\infty$ and $\{x_{m_j}\}_{j=1}^\infty$ that converge to a and b and $a\neq b$, respectively. We construct a new subsequence $\{y_\ell\}_{\ell=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, as follows. Let $k_1=1$ and $y_1=x_{n_{k_1}}$. Let j_1 be the smallest integer so that $m_{j_1}>n_{k_1}$, and define

$y_2 = x_{m_{j_1}}$. Let k_2 be the smallest integer so that $n_{k_2} > m_{j_1}$, and define $y_3 = x_{n_{\ell_2}}$. We continue this process and obtain a sequence $\{y_\ell\}_{\ell=1}^\infty$ satisfying that

$$y_\ell = \begin{cases} y_{n_{k_{\frac{\ell+1}{2}}}} & \ell \text{ is odd}, \\ y_{m_{j_{\frac{\ell}{2}}}} & \ell \text{ is even}, \end{cases}$$

where $k_1, k_2, \cdots$ and $j_1, j_2, \cdots$ satisfy that $k_1 = 1$,

$$j_r = \min\{j \in \mathbb{N} \,|\, m_j > k_r\} \qquad \text{and} \qquad k_{r+1} = \min\{k \in \mathbb{N} \,|\, n_k > m_{j_r}\} \qquad \forall\, r \in \mathbb{N}.$$

Then $\{y_{2\ell-1}\}_{\ell=1}^\infty$, the collection of odd terms of $\{y_\ell\}_{\ell=1}^\infty$, is a subsequence of $\{x_{n_k}\}_{k=1}^\infty$ and $\{y_{2\ell}\}_{\ell=1}^\infty$, the collection of even terms of $\{y_\ell\}_{\ell=1}^\infty$, is a subsequence of $\{x_{m_j}\}_{j=1}^\infty$, and $\{y_{2\ell-1}\}_{\ell=1}^\infty$ converges to a while $\{y_{2\ell}\}_{\ell=1}^\infty$ converges to b, and $a \neq b$. By a Proposition we talked about in class, $\{y_\ell\}_{\ell=1}^\infty$ does not converges, a contradiction. □

Problem 4. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $f : \mathbb{F} \to \mathbb{F}$ be a function so that

$$|f(x) - f(y)| \leqslant \alpha |x - y| \qquad \forall\, x, y \in \mathbb{F},$$

where $\alpha \in \mathbb{F}$ is a constant satisfying $0 < \alpha < 1$. Pick an arbitrary $x_1 \in \mathbb{F}$, and define $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$.

Proof. First we claim that if $0 < \alpha < 1$, then $\lim\limits_{n\to\infty} \alpha^n = 0$. In fact, we have $\dfrac{1}{\alpha} > 1$; thus by the fact that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ (which is from Archimedean property), there exists $p > 0$ such that

$$1 + \frac{1}{p} < \frac{1}{\alpha}.$$

Therefore,

$$\frac{1}{\alpha^p} > \Big(1 + \frac{1}{p}\Big)^p \geqslant 1 + C^p_1 \frac{1}{p} = 2$$

which implies that

$$0 < \alpha^p < \frac{1}{2}.$$

By the fact that $2^n \geqslant n$ for all $n \geqslant \mathbb{N}$ (which can be shown by induction), we find from the Sandwich Lemma that

$$\lim_{n\to\infty} \alpha^{pn} = 0.$$

Let $\varepsilon > 0$ be given. The identity above shows the existence of $N_1 > 0$ such that $|\alpha^{pn}| < \varepsilon$ whenever $n \geqslant N_1$. Let $N = pN_1$. Then if $n \geqslant N$,

$$|\alpha^n| \leqslant |\alpha^{pN_1}| < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} \alpha^n = 0$.

Next by the fact that $|f(x) - f(y)| \leqslant \alpha |x - y|$ and $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leqslant \alpha |x_n - x_{n-1}| \qquad \forall\, n \geqslant 2;$$

thus

$$|x_{n+1} - x_n| \leqslant \alpha |x_n - x_{n-1}| \overset{\text{(if } n \geqslant 3)}{\leqslant} \alpha^2 |x_{n-1} - x_{n-2}| \leqslant \cdots \leqslant \alpha^{n-1} |x_2 - x_1| \,.$$

Therefore, if $n > m$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leqslant \alpha^{n-2}|x_2 - x_1| + \alpha^{n-3}|x_2 - x_1| + \cdots + \alpha^{m-1}|x_2 - x_1| \\
&= \big(\alpha^{n-2} + \alpha^{n-3} + \alpha^{m-1}\big)|x_2 - x_1| \leqslant \frac{\alpha^{m-1}}{1-\alpha}|x_2 - x_1| \,.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \alpha^n = 0$, there exists $N > 0$ such that

$$\frac{\alpha^{n-1}}{1-\alpha}|x_2 - x_1| < \varepsilon \quad \text{whenever} \quad n \geqslant N \,.$$

Then if $n > m \geqslant N$, by the fact that $|x_n - x_m| \leqslant \dfrac{\alpha^{m-1}}{1-\alpha}|x_2 - x_1|$ we obtain that $|x_n - x_m| < \varepsilon$. □