# Exercise Problem Sets 3

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The first problem in this exercise set is a continuation of Problem 1 in Exercise 2.

**Problem 1.** Complete the following.

1. Verify the Wallis's formula: if $n$ is a non-negative integer, then
$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \frac{(2^n n!)^2}{(2n+1)!}$$
and
$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\,.$$

2. Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Show that $\displaystyle\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

3. Let $s_n = \dfrac{n!}{n^{n+0.5}e^{-n}}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence; that is, $s_n \geqslant s_{n+1}$ for all $n \in \mathbb{N}$.

4. Suppose that you know that $\mathbb{R}$ satisfies **MSP**. Then $\displaystyle\lim_{n\to\infty} s_n$ exists due to **MSP**. Find the limit of $\{s_n\}_{n=1}^{\infty}$.

**Hint**:

2. Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$ for all $n \in \mathbb{N}$ and then apply the Sandwich lemma.

3. Consider the function $f(x) = \left(1 + \frac{1}{x}\right)^{x+0.5}$.

*Proof.* 4. Since $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence and is bounded from below. By the monotone sequence property, $\displaystyle\lim_{n\to\infty} s_n = s$ exists. Note that
$$\begin{aligned}\frac{I_{2n+1}}{I_{2n}} &= \frac{2}{\pi}\frac{(2^n n!)^4}{(2n)!(2n+1)!} = \frac{2^{4n+1}}{\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(n^{n+0.5}e^{-n})^4}{(2n)^{2n+0.5}e^{-2n}(2n+1)^{2n+1.5}e^{-(2n+1)}}\\ &= \frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(2n)^{2n+1.5}}{(2n+1)^{2n+1.5}} = \frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\Big(1 + \frac{1}{2n}\Big)^{-2n-1.5}.\end{aligned}$$
Therefore, 2 implies that
$$1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{e}{2\pi}\frac{s_n^4}{s_{2n}s_{2n+1}}\frac{(2n)^{2n+1.5}}{(2n+1)^{2n+1.5}} = \frac{e}{2\pi}s^2\lim_{n\to\infty}\Big(1+\frac{1}{2n}\Big)^{-2n-1.5} = \frac{s^2}{2\pi}\,;$$
thus $s = \sqrt{2\pi}$ (since $s_n \geqslant 0$). □

**Problem 2.** Let $(\mathbb{F}, +, \cdot, \leqslant)$ an ordered field satisfying the monotone sequence property, and $y \in \mathbb{F}$ satisfying $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt{y}$ in the last example in class).

2. Show that $y^n - 1 > n(y-1)$ for all $n \in \mathbb{N}\backslash\{1\}$; thus $y - 1 > n(y^{1/n} - 1)$.

3. Show that if $t > 1$ and $n > (y-1)/(t-1)$, then $y^{1/n} < t$.

4. Show that $\lim\limits_{n\to\infty} y^{1/n} = 1$ as $n \to \infty$.

*Proof.* 1. For each $k \in \mathbb{N}$, let $N_k$ be the largest integer satisfying that $\big(\frac{N_k}{n^k}\big)^n \leqslant y$ but $\big(\frac{N_k+1}{n^k}\big)^n > y$. Define $x_k = \dfrac{N_k}{n^k}$. Then

(a) By binomial expansion, for each $k \in \mathbb{N}$ we have

$$x_k^n \leqslant y < 1 + C^n_1 y + C^n_2 y^2 + \cdots + C^n_n y^n = (1+y)^n\,;$$

thus Problem **??** implies that $x_k < 1 + y$. Therefore, $\{x_k\}_{k=1}^\infty$ is bounded from above.

(b) For each $k \in \mathbb{N}$, $\big(\frac{nN_k}{n^{k+1}}\big)^n = \big(\frac{N_k}{n^k}\big)^n \leqslant y$; thus $N_{k+1} \geqslant nN_k$. Therefore, for each $k \in \mathbb{N}$,

$$x_k = \frac{N_k}{n^k} = \frac{nN_k}{n^{k+1}} \leqslant \frac{N_{k+1}}{n^{k+1}} = x_{k+1}$$

which shows that $\{x_k\}_{k=1}^\infty$ is increasing.

Therefore, **MSP** implies that $\{x_k\}_{k=1}^\infty$ converges. Assume that $x_k \to x$ as $k \to \infty$ for some $x \in \mathbb{F}$. Then the fact that $x_k^n \leqslant y$ for all $k \in \mathbb{N}$ implies that $x^n \leqslant y$. On the other hand,

$$\Big(x_k + \frac{1}{n^k}\Big)^n \geqslant y \qquad \forall\, k \in \mathbb{N}\,;$$

thus **AP** (a consequence of **MSP**) implies that

$$x^n = \Big(\lim_{k\to\infty} x_k + \lim_{k\to\infty}\frac{1}{n^k}\Big)^n = \lim_{k\to\infty}\Big(x_k + \frac{1}{n^k}\Big)^n \geqslant y\,.$$

Therefore, $x^n = y$. Problem 2 then shows that there is only one $x > 0$ satisfying $x^n = y$. This $x$ will be denoted by $y^{\frac{1}{n}}$.

2. For $y > 1$, let $z = y - 1$. Then $z > 0$ so that for $n > 1$, the binomial expansion shows that

$$\begin{aligned} y^n - 1 &= (1+z)^n - 1 = 1 + C^n_1 z + C^n_2 z^2 + \cdots + C^n_n z^n - 1 = C^n_1 z + C^n_2 z^2 + \cdots + C^n_n z^n \\ &> nz = n(y-1)\,. \end{aligned}$$

Therefore, replacing $y$ by $y^{\frac{1}{n}}$ in the inequality above, we conclude that

$$y - 1 > n(y^{\frac{1}{n}} - 1) \qquad \forall\, n \in \mathbb{N}\backslash\{1\}\,.$$

3. Suppose that $y^{\frac{1}{n}} \geqslant t > 1$. Then 2 implies that for $n \in \mathbb{N}\backslash\{1\}$,

$$y - 1 > n(y^{\frac{1}{n}} - 1) \geqslant n(t-1)\,.$$

Therefore, $n \leqslant \dfrac{y-1}{t-1}$, a contradiction.

4. Let $k \in \mathbb{N}$ and $t = 1 + \frac{1}{k}$ in 3. Then for $n > k(y-1)$,

$$1 \leqslant y^{\frac{1}{n}} < 1 + \frac{1}{k}.$$

Since $n \to \infty$ as $k \to \infty$, by the Sandwich Lemma we conclude that $\lim_{n\to\infty} y^{\frac{1}{n}} = 1$. □

**Problem 3.** Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and $S \subseteq \mathbb{F}$ be bounded below and non-empty. Show that

$$\inf S = \sup\{x \in \mathbb{F} \,|\, x \text{ is a lower bound for } S\}$$

and

$$\sup S = \inf\{x \in \mathbb{F} \,|\, x \text{ is an upper bound for } S\}.$$

*Proof.* Define $A = \{x \in \mathbb{F} \,|\, x \text{ is a lower bound for } S\}$. Since $S$ is non-empty, every element in $S$ is an upper bound for $A$; thus $A$ is bounded from above. By the least upper bound property, $b = \sup A \in \mathbb{F}$ exists. Note that by the definition of $A$,

$$\text{if } x \in A, \text{ then } x \leqslant s \text{ for all } s \in S. \tag{$\star$}$$

Let $\varepsilon > 0$ be given. Then $b - \varepsilon$ is not an upper bound for $A$; thus there exists $x \in A$ such that $b - \varepsilon < x$. Then $(\star)$ implies that $b - \varepsilon < s$ for all $s \in S$. Since $\varepsilon > 0$ is given arbitrarily, $b \leqslant s$ for all $s \in S$; thus $b$ is a lower bound for $S$.

Suppose that $b$ is not the greatest lower bound for $S$. There exists $m > b$ such that $m \leqslant s$ for all $s \in S$. Therefore, $m \in A$; thus $m \leqslant b$, a contradiction. □

**Problem 4.** Let $A, B$ be two sets, and $f : A \times B \to \mathbb{F}$ be a function, where $(\mathbb{F}, +, \cdot, \leqslant)$ is an ordered field satisfying the least upper bound property. Show that

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B}\Big(\sup_{x\in A} f(x,y)\Big) = \sup_{x\in A}\Big(\sup_{y\in B} f(x,y)\Big).$$

*Proof.* Note that

$$f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, (x,y) \in A \times B;$$

thus

$$\sup_{x\in A} f(x,y) \leqslant \sup_{(x,y)\in A\times B} f(x,y) \qquad \forall\, y \in B.$$

The inequality above further shows that

$$\sup_{y\in B}\Big(\sup_{x\in A} f(x,y)\Big) \leqslant \sup_{(x,y)\in A\times B} f(x,y). \tag{$\star$}$$

Now we show the reverse inequality.

1. Suppose that $\sup\limits_{(x,y)\in A\times B} f(x,y) = M < \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that

$$f(x_k, y_k) > M - \frac{1}{k}.$$

Therefore,

$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{x\in A} f(x, y_k)$$

which further implies that

$$M - \frac{1}{k} < f(x_k, y_k) \leqslant \sup_{y\in B}\big(\sup_{x\in A} f(x,y)\big).$$

Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\sup\limits_{y\in B}\big(\sup\limits_{x\in A} f(x,y)\big) \geqslant M$.

2. Suppose that $\sup\limits_{(x,y)\in A\times B} f(x,y) = \infty$. Then for each $k \in \mathbb{N}$, there exists $(x_k, y_k) \in A \times B$ such that

$$f(x_k, y_k) > k.$$

Therefore,

$$k < f(x_k, y_k) \leqslant \sup_{x\in A} f(x, y_k)$$

which further implies that

$$k < f(x_k, y_k) \leqslant \sup_{y\in B}\big(\sup_{x\in A} f(x,y)\big).$$

Since the inequality above holds for all $k \in \mathbb{N}$, we find that $\sup\limits_{y\in B}\big(\sup\limits_{x\in A} f(x,y)\big) = \infty$.

With the help of $(\star)$, we conclude that $\sup\limits_{(x,y)\in A\times B} f(x,y) = \sup\limits_{y\in B}\big(\sup\limits_{x\in A} f(x,y)\big)$. □

**Problem 5.** Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the least upper bound property, and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$. Define

$$\|\boldsymbol{x}\|_1 = \sum_{k=1}^{n} |x_k| \qquad \text{and} \qquad \|\boldsymbol{x}\|_\infty = \max\big\{|x_1|, |x_2|, \cdots, |x_n|\big\}.$$

Show that

1. $\|\boldsymbol{x}\|_1 = \sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\Big\}$.
2. $\|\boldsymbol{y}\|_\infty = \sup\Big\{\sum\limits_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\Big\}$.

*Proof.* Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ be given. Then

$$\sum_{k=1}^{n} x_k y_k \leqslant \sum_{k=1}^{n} |x_k||y_k| \leqslant \sum_{k=1}^{n} |x_k|\|\boldsymbol{y}\|_\infty = \|\boldsymbol{y}\|_\infty \sum_{k=1}^{n} |x_k| = \|\boldsymbol{y}\|_\infty \|\boldsymbol{x}\|_1.$$

Therefore,

$$\sup\Big\{\sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{y}\|_\infty = 1\Big\} \leqslant \|\boldsymbol{x}\|_1 \qquad \text{and} \qquad \sup\Big\{\sum_{k=1}^{n} x_k y_k \,\Big|\, \|\boldsymbol{x}\|_1 = 1\Big\} \leqslant \|\boldsymbol{y}\|_\infty.$$

Next we show that the two inequalities are in fact equalities by showing that the right-hand side of the inequalities belongs to the sets (this is because if $b \in A$ is an upper bound for $A$, then $b$ is the least upper bound for $A$).

1. $\sup\left\{ \sum_{k=1}^{n} x_k y_k \,\middle|\, \|\boldsymbol{y}\|_\infty = 1 \right\} = \|\boldsymbol{x}\|_1$: W.L.O.G. we can assume that $\boldsymbol{x} \neq \mathbf{0}$. For a given $\boldsymbol{x} \in \mathbb{F}^n$, define $y_k \in \mathbb{F}$ by

$$y_k = \begin{cases} \dfrac{\overline{x_k}}{|x_k|} & \text{if } x_k \neq 0\,, \\ 0 & \text{if } x_k = 0\,, \end{cases}$$

where $\overline{x_k}$ denotes the complex conjugate of $x_k$. Then $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ satisfies $\|\boldsymbol{y}\|_\infty = 1$ (since at least one component of $\boldsymbol{x}$ is non-zero), and

$$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} |x_k| = \|\boldsymbol{x}\|_1\,.$$

2. $\sup\left\{ \sum_{k=1}^{n} x_k y_k \,\middle|\, \|\boldsymbol{x}\|_1 = 1 \right\} = \|\boldsymbol{y}\|_\infty$: W.L.O.G. we can assume that $\boldsymbol{y} \neq \mathbf{0}$. Suppose that $\|\boldsymbol{y}\|_\infty = |y_m| \neq 0$ for some $1 \leqslant m \leqslant n$; that is, the maximum of the absolute value of components occurs at the $m$-th component. Define $x_j \in \mathbb{F}$ by

$$x_j = \begin{cases} \dfrac{\overline{y_m}}{|y_m|} & \text{if } j = m\,, \\ 0 & \text{if } j \neq m\,, \end{cases}$$

where $\overline{y_m}$ is the complex conjugate of $y_m$. Then $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ satisfies $\|\boldsymbol{x}\|_1 = 1$ (since only one component of $\boldsymbol{x}$ is non-zero), and

$$\sum_{k=1}^{n} x_k y_k = \frac{\overline{y_m}}{|y_m|} y_m = |y_m| = \|\boldsymbol{y}\|_\infty\,. \qquad \square$$