Exercise Problem Sets 1

Sept. 17. 2021

Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b \in \mathbb{F}$. Show that $a \leqslant b$ if and only if for all $\varepsilon > 0$, $a < b + \varepsilon$.

Proof. The direction "⇒" is trivial, so we only prove the direction "⇐". Suppose the contrary that $a > b$. Let $\varepsilon = a - b$. Then $\varepsilon > 0$; thus

$$a < b + (a - b) = a\,,$$

a contradiction. □

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $x, y \in \mathbb{F}$, and $n \in \mathbb{N}$. Show that

1. If $0 \leqslant x < y$, then $x^n < y^n$.
2. If $0 \leqslant x, y$ and $x^n < y^n$, then $x < y$.

Proof. 1. Let $S = \{n \in \mathbb{N} \,|\, x^n < y^n\}$. Then $1 \in S$ by assumption. Suppose that $n \in S$. Then $0 \leqslant x^n < y^n$. By the fact that $0 \leqslant x < y$, we find that

$$x^{n+1} = x^n \cdot x < x^n \cdot y < y^n \cdot y = y^{n+1}\,;$$

thus $n + 1 \in S$. By induction, we conclude that $S = \mathbb{N}$.

2. Suppose the contrary that $x \geqslant y$. Then 1 implies that $x^n \geqslant y^n$, a contradiction. □