Exercise Problem Sets 10

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Problem 1. Let $f, g \in \mathscr{S}(\mathbb{R}^n)$. Show that

1. $\mathscr{F}(f \star g) = \widehat{f} \cdot \widehat{g}$. 2. $\mathscr{F}^*(f \star g) = \check{f} \cdot \check{g}$. 3. $\mathscr{F}(fg) = \widehat{f} \star \widehat{g}$. 4. $\mathscr{F}^*(fg) = \check{f} \star \check{g}$.

Proof. 1. See the proof of Theorem 9.25 in the lecture note.

2. Replacing $e^{-ix\cdot\xi}$ by $e^{ix\xi}$ in the proof of Theorem 9.25 in the lecture note provides a proof of $\mathscr{F}^*(f \star g) = \check{f} \cdot \check{g}$. One can also prove as follows. Let $\tilde{}$ be the reflection operator. Then

$$\mathscr{F}^*(f \star g)(\xi) = \mathscr{F}(f \star g)(-\xi) = \widehat{f}(-\xi) \cdot \widehat{g}(-\xi) = \tilde{\widehat{f}}(\xi) \cdot \tilde{\widehat{g}}(\xi) = \check{f}(\xi) \cdot \check{g}(\xi)\,.$$

3. By 2, $\mathscr{F}^*(\widehat{f} \star \widehat{g}) = \check{\widehat{f}} \cdot \check{\widehat{g}} = f \cdot g$; thus by the Fourier inversion formula,

$$\widehat{f} \star \widehat{g} = \mathscr{F}\mathscr{F}^*(\widehat{f} \star \widehat{g}) = \mathscr{F}(f \cdot g)\,.$$

4. By 1, $\mathscr{F}(\check{f} \star \check{g}) = \widehat{\check{f}} \cdot \widehat{\check{g}} = f \cdot g$; thus by the Fourier inversion formula,

$$\check{f} \star \check{g} = \mathscr{F}^*\mathscr{F}(\check{f} \star \check{g}) = \mathscr{F}^*(f \cdot g)\,.$$

□

Problem 2. Find the Fourier transform of the following functions.

1. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = xe^{-tx^2}$ for $t > 0$.

2. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \chi_{(-a,a)}(x)$, the characteristic (indicator) function of the set $(-a, a)$.

3. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} e^{-tx} & \text{if } x > 0\,, \\ 0 & \text{if } x \leqslant 0\,, \end{cases}$ where $t > 0$.

Solution. 1. Integrating by parts,

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-tx^2} e^{-ix\xi}\,dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} xe^{-tx^2} e^{-ix\xi}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \Big[\frac{-1}{2t} e^{-tx^2} e^{-ix\xi}\Big|_{x=-R}^{x=R} - \frac{i\xi}{2t} \int_{-R}^{R} e^{-tx^2} e^{-ix\xi}\,dx\Big] \\ &= -\frac{i\xi}{2t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-tx^2} e^{-ix\xi}\,dx = -\frac{i\xi}{2t} \mathscr{F}_x\big[e^{-tx^2}\big](\xi)\,.\end{aligned}$$

Noting that with $P_t(x) = \dfrac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}}$, the formula $\widehat{P}_t(\xi) = e^{-\frac{1}{2}t\xi^2}$ implies that

$$\mathscr{F}_x\big[e^{-tx^2}\big](\xi) = \frac{1}{\sqrt{2t}} \widehat{P_{\frac{1}{2t}}}(\xi) = \frac{1}{\sqrt{2t}} e^{-\frac{\xi^2}{4t}}\,;$$

thus $\widehat{f}(\xi) = -\dfrac{i\xi}{\sqrt{2t}^3} e^{-\frac{\xi^2}{4t}}$.

2. We integrate directly and obtain that if $\xi \neq 0$

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \big[\cos(x\xi) - i\sin(x\xi)\big]\, dx \\ &= \frac{1}{\sqrt{2\pi}\xi}\big[\sin(x\xi) + i\cos(x\xi)\big]\Big|_{x=-a}^{x=a} = \frac{2\sin(a\xi)}{\sqrt{2\pi}\xi}\,,\end{aligned}$$

while $\widehat{f}(0) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-a}^{a} 1\, dx = \dfrac{2a}{\sqrt{2\pi}}$. Therefore,

$$\widehat{f}(\xi) = \begin{cases} \dfrac{2\sin(a\xi)}{\sqrt{2\pi}\xi} & \text{if } \xi \neq 0\,, \\ \dfrac{2a}{\sqrt{2\pi}} & \text{if } \xi = 0\,. \end{cases}$$

3. Since $t > 0$, $\lim\limits_{R\to\infty} e^{-(t+i\xi)R} = 0$ for all $\xi \in \mathbb{R}$; thus

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tx} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_0^R e^{-(t+i\xi)x}\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \frac{e^{-(t+i\xi)x}}{-(t+i\xi)}\Big|_{x=0}^{x=R} \\ &= \frac{1}{\sqrt{2\pi}(t+i\xi)}\end{aligned}$$

□

Problem 3. A vector-valued function $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$ is called a Schwartz function, still denoted by $\boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)$, if $u_j \in \mathscr{S}(\mathbb{R}^n)$ for all $1 \leqslant j \leqslant n$. Show the Korn inequality

$$\sum_{i,j=1}^n \big\|\epsilon_{ij}(\boldsymbol{u})\big\|^2_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{2} \sum_{i,j=1}^n \Big\|\frac{\partial u_j}{\partial x_i}\Big\|^2_{L^2(\mathbb{R}^n)} \qquad \forall\, \boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n)\,,$$

where $\epsilon_{ij}(\boldsymbol{u}) = \dfrac{1}{2}\Big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\Big)$ is the symmetric part of $D\boldsymbol{u}$.

Hint: Use Lemma 9.11 in the lecture note and the Plancherel formula.

Proof. By the Plancherel formula,

$$\begin{aligned}&\big\|\epsilon_{ij}(u)\big\|^2_{L^2(\mathbb{R}^n)} \\ &\quad= \frac{1}{4}\sum_{i,j=1}^n \int_{\mathbb{R}^n} \big[\xi_i\xi_i\widehat{u}_j(\xi)\overline{\widehat{u}_j(\xi)} + \xi_j\xi_j\widehat{u}_i(\xi)\overline{\widehat{u}_i(\xi)} + \xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)} + +\xi_j\xi_i\overline{\widehat{u}_i(\xi)}\widehat{u}_j(\xi)\big]\,d\xi \\ &\quad= \sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 + 2\xi_j\xi_i\widehat{u}_i(\xi)\overline{\widehat{u}_j(\xi)}\big]\,d\xi \\ &\quad\geqslant \sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2 - \xi_i^2|\widehat{u}_i(\xi)|^2 - \xi_j^2|\widehat{u}_j(\xi)|^2\big]\,d\xi \\ &\quad\geqslant \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_i(\xi)|^2 d\xi + \frac{1}{4}\sum_{i\neq j}\int_{\mathbb{R}^n} \big[\xi_i^2|\widehat{u}_j(\xi)|^2 + \xi_j^2|\widehat{u}_i(\xi)|^2\big]\,d\xi \\ &\quad\geqslant \frac{1}{2}\sum_{i,j=1}^n \int_{\mathbb{R}^n} \xi_i^2|\widehat{u}_j(\xi)|^2 d\xi = \frac{1}{2}\sum_{i,j=1}^n \Big\|\frac{\partial u_j}{\partial x_i}\Big\|^2_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

□

Problem 4. 1. Let d_r denote the dilation operator defined by $d_r f(x) = f\big(\frac{x}{r}\big)$. Show that

$$\mathscr{F}(d_r f) = r^n d_{1/r}\mathscr{F}(f) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,.$$

2. In some occasions (especially in engineering applications), the Fourier transform and inverse Fourier transform of a (Schwartz) function f are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x\cdot\xi}dx \qquad \text{and} \qquad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{i2\pi x\cdot\xi}d\xi\,.$$

Show that under this definition, $\check{\widehat{f}} = \widehat{\check{f}} = f$ for all $f \in \mathscr{S}(\mathbb{R}^n)$. Note that you can use the Fourier Inversion Formula that we derive in class.

Proof. Let $\mathscr{F}$ denote the Fourier transform operator that we used in class, and $\widehat{\ }$ be the Fourier transform operator in this problem.

1. Let d_r denote the dilation operator define by $(d_r f)(x) = f(rx)$. By the change of variables formula,

$$\begin{aligned}\mathscr{F}(d_r f)(\xi) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n}(d_r f)(x)e^{-ix\cdot\xi}\,dx = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(r^{-1}x)e^{-ix\cdot\xi}\,dx\\ &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y)e^{-iry\cdot\xi}r^n\,dy = \frac{r^n}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(y)e^{-iy\cdot(r\xi)}\,dy\\ &= r^n\mathscr{F}(f)(r\xi) = r^n\big[d_{\frac{1}{r}}\mathscr{F}(f)\big](\xi)\,.\end{aligned}$$

Therefore, $\mathscr{F}(d_r f) = r^n d_{1/r}\mathscr{F}(f)$ for all $f \in \mathscr{S}(\mathbb{R}^n)$.

2. Replacing f by $d_{1/r}f$ in the equation established in 1, we find that

$$\mathscr{F}(f) = \mathscr{F}\big(d_r d_{\frac{1}{r}} f\big) = r^n d_{\frac{1}{r}}\mathscr{F}\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond$}$$

Similarly, $\mathscr{F}^*(d_r f) = r^n d_{\frac{1}{r}}\mathscr{F}^*(f)$ so that

$$\mathscr{F}^*(f) = r^n d_{\frac{1}{r}}\mathscr{F}^*\big(d_{\frac{1}{r}} f\big) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^n)\,. \tag{$\diamond\diamond$}$$

Note that

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx = \sqrt{2\pi}^n\mathscr{F}(f)(2\pi\xi) = \sqrt{2\pi}^n\big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi)\\ &= \frac{1}{\sqrt{2\pi}^n}(2\pi)^n\big[d_{\frac{1}{2\pi}}\mathscr{F}(f)\big](\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi}f)(\xi)\end{aligned}$$

and

$$\check{f}(\xi) = \widehat{f}(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}(d_{2\pi}f)(-\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi}f)(\xi)\,.$$

Therefore, $(\diamond)$ implies that

$$\begin{aligned}\check{\widehat{f}}(\xi) &= \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*(d_{2\pi}\widehat{f})(\xi) = \frac{1}{\sqrt{2\pi}^n}\mathscr{F}^*\Big(\frac{1}{\sqrt{2\pi}^n}d_{2\pi}\mathscr{F}(d_{2\pi}f)\Big)(\xi)\\ &= \mathscr{F}^*\big((2\pi)^{-n}d_{2\pi}\mathscr{F}(d_{2\pi}f)\big)(\xi) = \mathscr{F}^*(\mathscr{F}f)(\xi) = f(\xi)\,.\end{aligned}$$

Similarly, $(\diamond\diamond)$ implies that

$$\widehat{\check{f}}(\xi) = \mathscr{F}\big((2\pi)^{-n}d_{2\pi}\mathscr{F}^*(d_{2\pi}f)\big)(\xi) = \mathscr{F}(\mathscr{F}^*f)(\xi) = f(\xi)\,.$$

□